ON THE GEOMETRY OF A PROPOSED CURVE COMPLEX ANALOGUE FOR $\mathrm{Out}(F_n)$

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Abstract. The group $\mathrm{Out}(F_n)$ of outer automorphisms of the free group has been an object of active study for many years, yet its geometry is not well understood. Recently, effort has been focused on finding a hyperbolic complex on which $\mathrm{Out}(F_n)$ acts, in analogy with the curve complex for the mapping class group. Here, we focus on one of these proposed analogues: the edge splitting complex $\mathcal{ES}_n$, equivalently known as the separating sphere complex. We characterize geodesic paths in its 1-skeleton $\mathcal{ES}_n^1$ algebraically, and use our characterization to find lower bounds on distances between points in this graph.

Our distance calculations allow us to find quasiflats of arbitrary dimension in $\mathcal{ES}_n$. This shows that $\mathcal{ES}_n$: is not hyperbolic, has infinite asymptotic dimension, and is such that every asymptotic cone is infinite dimensional. These quasiflats contain an unbounded orbit of a reducible element of $\mathrm{Out}(F_n)$. As a consequence, there is no coarsely $\mathrm{Out}(F_n)$-equivariant quasiisometry between $\mathcal{ES}_n$ and other proposed curve complex analogues, including the regular free splitting complex $\mathcal{FS}_n$, the (nontrivial intersection) free factorization complex $\mathcal{FF}_n$, and the free factor complex $\mathcal{F}_n$, leaving hope that some of these complexes are hyperbolic.

1. Introduction

Let $\mathrm{Out}(F_n)$ denote the group of outer automorphisms of the free group F_n of rank n, where we assume throughout this paper that $n > 2$. We wish to study the geometry of $\mathrm{Out}(F_n)$, by examining the geometry of certain spaces on which group acts. There is a strong analogy between $\mathrm{Out}(F_n)$ and the mapping class group of a surface on the one hand and arithmetic groups on the other, which has been pursued quite fruitfully in the last couple of decades. This approach began in earnest with the foundational paper of Culler and Vogtmann [CV86], which introduced Outer Space, the analogue for $\mathrm{Out}(F_n)$ of Teichmüller space for the mapping class group and of symmetric spaces for arithmetic groups. The work that followed has yielded numerous statements about the topological, homological, and cohomological properties of $\mathrm{Out}(F_n)$ and the spaces it acts upon – see for instance [Vog02] for an excellent survey.

While the topology of Outer Space is well-understood, its geometry is not. In contrast, the geometries of Teichmüller space and the symmetric spaces are well-studied. One key ingredient for the study of Teichmüller space is the celebrated result of Masur and Minsky, who proved that the curve complex is hyperbolic [MM99]. The *curve complex* is the complex whose vertex set is the set of isotopy classes of simple closed curves on the surface, and where a k-simplex corresponds to $k+1$ isotopy classes which have representatives that are disjoint. Moreover, there is a 'nice' map from Teichmüller space to the curve complex, so that the hyperbolicity of the curve complex has led to many further statements on the geometry of Teichmüller space and the mapping class group [BKMM10]. The curve complex has been used, for instance, to prove quasiisometric rigidity of the mapping class group. The analogous key ingredients in the study of arithmetic groups are Tits buildings, which again yield, for instance, rigidity theorems. The 'correct' analogue for $\mathrm{Out}(F_n)$ is still unknown, and much recent effort has been directed towards finding one – in particular, one which is hyperbolic.

There are many possible ways of defining such an analogue. We will formally define the most relevant two soon, but we leave definitions of the remaining complexes and graphs to the references. Before we list some of proposed analogues, let us mention that in most cases they are defined as complexes, but for our purposes (detecting hyperbolicity and distinguishing the spaces up to quasiisometry) it is enough to consider just 1-skeletons of the complexes. For each complex we will denote its 1-skeleton by adding superscript '1' to the notation of the complex. Although we

will rigorously define and work only with 1-skeletons of the complexes to simplify exposition, our results apply to the corresponding complexes as well.

Complexes and graphs which deserve mention as possible analogues include: the *sphere complex* [Hat95], also called the *free splitting complex* $\mathcal{FS}_n$, and its 1-skeleton $\mathcal{FS}^1_n$, called the *free splitting graph* [AS09]; the *(common refinement) free factorization complex*, defined in [HV98a] for $\mathrm{Aut}(F_n)$, whose $\mathrm{Out}(F_n)$ version we call the *edge splitting complex* $\mathcal{ES}_n$ in this paper; the *free factor complex* $\mathcal{F}_n$ (also defined initially for $\mathrm{Aut}(F_n)$ in [HV98b]); and the *intersection graph* of Kapovich and Lustig [KL09]. Kapovich and Lustig [KL09] in fact list 9 graphs which could be an analogue of the curve complex. They include the 1-skeleton of the edge splitting complex which we call the *edge splitting graph* $\mathcal{ES}^1_n$ (called the *free splitting graph* in [KL09], though they do not allow HNN-extensions as vertices) and the 1-skeleton of the free factor complex which we call the *free factor graph* $\mathcal{F}^1_n$ (called the *dominance graph* in [KL09]).

Kapovich and Lustig claim that, among the 9 graphs they list, there are at most 3 quasiisometry classes. Representatives of the three mentioned quasiisometry classes are the edge splitting graph, the free factor graph, and the intersection graph. We intend to show that the class containing the edge splitting graph cannot be coarsely $\mathrm{Out}(F_n)$-equivariantly quasiisometric to the other two. For our purposes, it will be more convenient to use what we call the *(nontrivial intersection) free factorization graph* $\mathcal{FF}^1_n$ instead of the free factor graph as a representative of the second quasiisometry class. Note that our free factorization graph is not the 1-skeleton of Hatcher and Vogtmann's (common refinement) free factorization complex (herein called the edge splitting graph), and that this graph was called the *dual free splitting graph* in [KL09], though again in the latter reference they did not allow HNN-extensions as vertices. We now define the edge splitting graph and the free factorization graph.

Definition 1.1 ($\mathcal{ES}^1_n$ and $\mathcal{FF}^1_n$)**.** For $n > 2$, define the *edge splitting graph*, denoted $\mathcal{ES}^1_n$, to be the graph whose vertices correspond to conjugacy classes $[\langle x_1, \dots, x_k\rangle * \langle x_{k+1}, \dots, x_n\rangle]$ of free factorizations $\langle x_1, \dots, x_k\rangle * \langle x_{k+1}, \dots, x_n\rangle$ of F_n into two nontrivial free factors. Two vertices of $\mathcal{ES}^1_n$ are connected with an edge if there exists a free factorization in each conjugacy class such that the two factorizations have a common refinement which is a free factorization into three nontrivial factors.

The *(nontrivial intersection) free factorization graph* $\mathcal{FF}^1_n$ has the same vertex set as $\mathcal{ES}^1_n$. Two vertices $[A * B]$ and $[C * D]$ are connected with an edge in $\mathcal{FF}^1_n$ if one of $A \cap C$, $A \cap D$, $B \cap C$, or $B \cap D$ is nontrivial.

The name of the edge splitting graph comes from Bass-Serre theory, where such a free factorization is a graph of groups decomposition of F_n with underlying graph having exactly two vertices and a single edge (with trivial edge group) between them. Note that the related *free splitting graph* $\mathcal{FS}^1_n$ (the 1-skeleton of the free splitting complex or equivalently the sphere complex) is defined similarly to $\mathcal{ES}^1_n$, but also allows conjugacy classes of splittings of F_n as HNN-extensions as vertices.

There are alternate ways to define each of these objects. In particular, the edge splitting graph $\mathcal{ES}^1_n$ is also known as the *separating sphere graph*, whose vertices are homotopy classes of separating essential embedded spheres in a 3-manifold with fundamental group F_n, and two vertices are adjacent if they have disjoint representatives. The free factorization graph can equivalently be defined in terms of Bass-Serre theory, where vertices are Bass-Serre trees of free splittings up to $\mathrm{Out}(F_n)$-equivariant isometry, and adjacency corresponds to having a common elliptic element.

Note there is a natural action of $\mathrm{Out}(F_n)$ on all of these spaces, where for $\mathcal{ES}^1_n$ and $\mathcal{FF}^1_n$ the action is induced by the action of $\mathrm{Out}(F_n)$ on free factorizations.

Not all that much is known about either of these graphs or their siblings. Hatcher showed that the sphere complex, which contains Outer Space as a dense subspace, is contractible (this gives an alternate proof of contractibility of Outer Space [CV86], as the contraction restricts to a contraction of Outer Space). Hatcher and Vogtmann showed that the edge splitting and free factor complexes – at least the $\mathrm{Aut}(F_n)$ versions of them, where we do not identify objects which differ by conjugation – are both $(n-2)$-spherical [HV98a, HV98b] (again, Hatcher and Vogtmann use the terminology 'free factorization complex' in place of 'edge splitting complex'). It seems

to be an open question whether the $\mathrm{Out}(F_n)$ versions of these complexes are also spherical. To study $\mathrm{Out}(F_n)$, Guirardel [Gui05] has introduced a notion of intersection form for two actions of $\mathrm{Out}(F_n)$ on $\mathbb{R}$-trees. Behrstock, Bestvina, and Clay [BBC09] used Guirardel's intersection form to describe the effect of applying *fully irreducible* automorphisms without periodic conjugacy classes to vertices in $\mathcal{ES}_n^1$. They also discuss the edge splitting complex (therein called the splitting complex though HNN extensions are not allowed, as in [KL09]), and a related complex called the *subgraph complex*. Kapovich and Lustig [Kap06, Lus04] have also introduced an intersection form (distinct from Guirardel's), inspired by the work of Bonahon [Bon91]. Kapovich and Lustig have shown that $\mathcal{ES}_n^1$ and $\mathcal{FF}_n^1$, as well as their intersection graph and 6 other related graphs, all have infinite diameter [KL09]. Recently, Yakov Berchenko-Kogan [BK10] characterized vertices of distance 2 apart in the *ellipticity graph*, a graph quasiisometric to $\mathcal{FF}_n^1$, using Stallings foldings. This effectively characterizes adjacent vertices in $\mathcal{FF}_n^1$. Very recently, Day and Putman [DP10] proved that another curve complex analogue, the *complex of partial bases*, is simply connected. The 1-skeleton of this complex is called the primitivity graph in [KL09], where it is also claimed that this graph is quasiisometric to the free factorization graph $\mathcal{FF}_n^1$. Aramayona and Souto have shown that $\mathrm{Out}(F_n)$ is precisely the group of simplicial automorphisms of the free splitting complex $\mathcal{FS}_n$ [AS09].

None of these spaces are yet known to be hyperbolic. Bestvina and Feign [BF10] have shown that, for any finite set S of *fully irreducible* outer automorphisms (see the paper for definitions), there exists a hyperbolic graph X with an isometric $\mathrm{Out}(F_n)$ action such that for any $\phi \in S$, ϕ acts with positive translation length on X. However, while these graphs are hyperbolic, they depend on the choice of the finite set S. The results of Bestvina and Feighn imply that for a fully irreducibly outer automorphism ϕ, the maps $\mathbb{Z} \to \mathcal{ES}_n^1$ and $\mathbb{Z} \to \mathcal{FF}_n^1$ given by $n \mapsto \phi^n v$ for any vertex v is a quasiisometric embedding. Behrstock, Bestvina, and Feighn [BBC09] state that "there is a hope that a proof of hyperbolicity of the curve complex generalizes to the [edge splitting] complex". However, we intend to prove:

Theorem 5.4. For $n > 2$, the space $\mathcal{ES}_n^1$ (and hence $\mathcal{ES}_n$) contains a quasiisometrically embedded copy of $\mathbb{R}^m$ for every $m \geq 1$.

Our proof relies on attaining an understanding of distances in $\mathcal{ES}_n^1$. To do so, we associate vertices of $\mathcal{ES}_n^1$ with bases of F_n. With this association, we are able to completely characterize (up to distance 4) the length of a path in $\mathcal{ES}_n^1$ via a simple algebraic notion which we call *number of index changes*. This characterization is made precise in Theorem 3.2 and the preceding discussion.

To utilize this translation from geometry to algebra, we then introduce an algebraic notion of complexity of a basis, which we call *i-length*. The notion of i-length is itself based roughly on having many subwords of elements of the basis with complicated Whitehead graphs. Our techniques, in turn, use a theorem of Stallings (see Section 4 for details). The bulk of this paper aims to translate this i-length notion of how complicated a basis is into a lower bound on distances between vertices in $\mathcal{ES}_n^1$, as shown in the following theorem:

Theorem 5.3. Let $\mathbf{x}$ be a basis of F_n, expressed in terms of a fixed standard basis $\mathbf{a}$. The distance between a vertex of $\mathcal{ES}_n^1$ associated to $\mathbf{a}$ and one associated to $\mathbf{x}$ is at least $\frac{|\mathbf{x}|_i}{24} - 1$, where $|\mathbf{x}|_i$ is the i-length of $\mathbf{x}$.

As immediate corollaries of Theorem 5.4, we obtain:

Corollary 5.5. The space $\mathcal{ES}_n^1$ is not Gromov hyperbolic.

In other words, $\mathcal{ES}_n^1$ is not the 'correct' curve complex analogue for $\mathrm{Out}(F_n)$. This shows that the 'hope' of [BBC09] is a false one, at least for the edge splitting graph. Indeed, it might be expected that the edge splitting graph is not hyperbolic: edge splittings correspond to separating spheres in the sphere complex. But in the mapping class group world, the subcomplex of the curve complex induced by only allowing separating curves is itself not hyperbolic [Sch06].

Corollary 5.6. The space $\mathcal{ES}_n^1$ has infinite asymptotic dimension. The dimension of every asymptotic cone of $\mathcal{ES}_n^1$ is infinite.

To the authors' knowledge, this is the only naturally defined space which has infinite asymptotic dimension and a natural cocompact group action of a group which is not known to have infinite asymptotic dimension. Thompson's group F acts on a cube complex with arbitrary-dimensional quasiflats [Far06], but has infinite asymptotic dimension (moreover, it is proved in [DS10] that F has exponential dimension growth). Via private communication, Moon Duchin claims that the Cayley graph of $\mathbb{Z}$ with respect to the infinite generating set consisting of powers of 2 has arbitrary-rank quasiflats. Thus, we have a group with finite asymptotic dimension acting on a space with infinite asymptotic dimension. However, this action is not cocompact: the quotient is a graph with one vertex and infinitely many edges. Both the mapping class group [BBK10] and arithmetic groups [Ji04] have finite asymptotic dimension, so the analogy between $\mathrm{Out}(F_n)$ and these groups suggests that $\mathrm{Out}(F_n)$ may in fact have finite asymptotic dimension.

There is a further interesting consequence of Theorem 5.4. There is a natural map id^* from $\mathcal{ES}_n^1$ to $\mathcal{FF}_n^1$ induced by the identity map on the vertex set. This map id^* is 1-Lipshitz: if two free factorizations have a common refinement, then any nontrivial elliptic element of the common refinement will have translation length 0 on both of the corresponding Bass-Serre trees. The quasiflats described in the proof of Theorem 5.4 are in fact such that, for every quasiflat, there exists a common elliptic element such that every vertex in that quasiflat has a representative where one factor contains the common elliptic element. Thus,

Corollary 5.7. The map $id^*\colon \mathcal{ES}_n^1 \to \mathcal{FF}_n^1$ is not a quasiisometry. Moreover, there is no coarsely $\mathrm{Out}(F_n)$-equivariant quasiisometry between $\mathcal{ES}_n^1$ and $\mathcal{FF}_n^1$.

An analogous results hold true for the relationships between the free factorization graph $\mathcal{ES}_n^1$ and the free factor graph $\mathcal{F}_n^1$ and between $\mathcal{ES}_n^1$ and the free splitting graph $\mathcal{FS}_n^1$. There is a natural (coarsely well-defined for $n > 2$) map $\Sigma\colon \mathcal{ES}_n^1 \to \mathcal{F}_n^1$ defined by sending a vertex $[A * B]$ in $\mathcal{ES}_n^1$ to the vertex $[A]$ in $\mathcal{F}_n^1$. Also there is a natural embedding $\imath\colon \mathcal{ES}_n^1 \to \mathcal{FS}_n^1$ defined by sending a vertex $[A * B]$ in $\mathcal{ES}_n^1$ to the vertex $[A * B]$ in $\mathcal{FS}_n^1$, which is quasisurjection. However, neither of the above maps is a quasiisometry:

Corollary 5.8. The maps $\Sigma\colon \mathcal{ES}_n^1 \to \mathcal{F}_n^1$ and $\imath\colon \mathcal{ES}_n^1 \to \mathcal{FS}_n^1$ are not quasiisometries. Moreover, there is no coarsely $\mathrm{Out}(F_n)$-equivariant quasiisometry between $\mathcal{ES}_n^1$ and $\mathcal{F}_n^1$, and between $\mathcal{ES}_n^1$ and $\mathcal{FS}_n^1$.

The last corollary provides a negative answer to a question of Bestvina and Feighn (the first half of Question 4.4 in [BF10]).

This paper is organized as follows. We begin in Section 2 by describing three ways of viewing an element of $\mathrm{Aut}(F_n)$. Being able to translate between these three perspectives will be useful at various points in the later proofs. In Section 3, we describe how to view vertices in $\mathcal{ES}_n^1$ and $\mathcal{FF}_n^1$ as pairs consisting of an element of $\mathrm{Aut}(F_n)$ and a proper nonempty subset of $\{1, 2, \ldots, n\}$ up to certain identifications. This viewpoint allows us to interpret distances in $\mathcal{ES}_n^1$ algebraically, in terms of elements of $\mathrm{Aut}(F_n)$, culminating in Theorem 3.2.

Most of the detail in the paper lies in Section 4. In this section, we introduce the notion of i-length. For technical reasons, we use three different notions of i-length: fixing some basis $\mathbf{a}$ of F_n, we have *simple* i-length for abstract words over $\mathbf{a}$, *conjugate reduced* i-length for subwords written over $\mathbf{a}$ of some other basis of F_n, and *full* i-length for bases of F_n themselves. In Section 4, we describe properties of each of these notions of i-length in turn. The section builds up to, and ends with, Theorem 5.3.

Finally in Section 5, we relate the algebraic notion of i-length to distances in $\mathcal{ES}_n^1$, and use this relationship to prove Theorem 5.4 and its corollaries, described above.

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2. Three interpretations of $\mathrm{Aut}(F_n)$

Fix a basis $\mathbf{a} = (a_1, \ldots, a_n)$ of F_n, considered as an ordered tuple. The group of all automorphisms of F_n has many interpretations. For our purposes, we will use three of these interpretations, as follows.

The first interpretation of $\mathrm{Aut}(F_n)$ is as in bijective correspondence with the set of ordered bases of F_n. Consider a basis $\mathbf{x} = (x_1, \ldots, x_n)$ of F_n as an ordered tuple. As $\mathbf{x}$ is a basis, there exists an automorphism $\phi_{\mathbf{x}}$ which maps $\mathbf{a}$ to $\mathbf{x}$; as automorphisms are uniquely specified by their action on a given generating set, $\phi_{\mathbf{x}}$ is unique. Thus, $\mathrm{Aut}(F_n)$ as a set is in bijective correspondence with the set

$$\mathbf{X} := \{\mathbf{x} = (x_1, \ldots, x_n) \in F_n^n \mid \mathbf{x} \text{ is an ordered basis}\}.$$

The second interpretation of $\mathrm{Aut}(F_n)$ is as products of *elementary Nielsen automorphisms*. Nielsen [Nie24] described a generating set for $\mathrm{Aut}(F_n)$ consisting of four types of generators:

Definition 2.1. An *elementary Nielsen automorphism* is an automorphism of F_n for which there exist indices i, j such that $i \neq j$, $a_k \mapsto a_k$ for $k \neq i, j$, and one of the following four possibilities holds:

(1) s_{ij} : $a_i \leftrightarrow a_j$
(2) t_i : $a_i \mapsto a_i^{-1}$
(3) a_{ij} : $a_i \mapsto a_i a_j$
(4) a_{ij}^{-1} : $a_i \mapsto a_i a_j^{-1}$

The group operation in $\mathrm{Aut}(F_n)$ with respect to Nielsen automorphisms is function composition, where automorphisms are composed as functions, right-to-left. Note a Nielsen automorphism ϕ acts on the Cayley graph of $\mathrm{Aut}(F_n)$ via the usual left action. We can interpret this action on the vertices of the Cayley graph in terms of the correspondence between $\mathrm{Aut}(F_n)$ and $\mathbf{X}$: an automorphism ϕ acting on a basis $\mathbf{x} \in \mathbf{X}$ has image $\phi(\mathbf{x}) = (\phi x_1, \ldots, \phi x_n) = \phi \circ \phi_{\mathbf{x}}(\mathbf{a})$.

The third interpretation of $\mathrm{Aut}(F_n)$ is as the group of Nielsen transformations. A *Nielsen transformation* is an action on the set of ordered bases of F_n (that is, on $\mathrm{Aut}(F_n)$, by the first interpretation) which may be decomposed as a product of *elementary Nielsen transformations*. These elementary Nielsen transformations are free-group analogues of the elementary row operations in $GL_n(\mathbb{Z}) = Aut(\mathbb{Z}^n)$, and, in fact, induce the elementary row operations under the abelianization map $F_n \to \mathbb{Z}^n$. There are four kinds of elementary Nielsen transformations:

Definition 2.2. An *elementary Nielsen transformation* is a map on the set of ordered bases $\mathbf{X} = \{\mathbf{x} = (x_1, \ldots, x_n)\}$ of F_n for which there exist indices i, j such that $i \neq j$, $x_k \mapsto x_k$ for $k \neq i, j$, and one of the following four possibilities hold:

(1) σ_{ij} : $x_i \leftrightarrow x_j$
(2) τ_i : $x_i \mapsto x_i^{-1}$
(3) α_{ij} : $x_i \mapsto x_i x_j$.
(4) α_{ij}^{-1} : $x_i \mapsto x_i x_j^{-1}$

An elementary Nielsen transformation of Types (3) and (4) are called *transvections*.

The group operation in $\mathrm{Aut}(F_n)$ with respect to Nielsen transformations is again composition, but transformations are composed left-to-right. Nielsen transformations act on $\mathbf{X}$ on the right.

The isomorphism between the groups generated by Nielsen automorphisms and by Nielsen transformations is clear: the isomorphism is $s_{ij} \mapsto \sigma_{ij}$, $t_i \mapsto \tau_i$, $a_{ij} \mapsto \alpha_{ij}$. Thus, a word in elementary Nielsen transformations may be considered as a word in Nielsen automorphisms, *written in the same order*, but with the order of composition reversed and the action on $\mathbf{X}$ on the left instead of the right.

These three interpretations are different aspects of the same concept: the set $\mathbf{X}$ may be viewed as the vertices of the Cayley graph of $\mathrm{Aut}(F_n)$; elementary Nielsen automorphisms form a generating set of $\mathrm{Aut}(F_n)$ and their action on $\mathbf{X}$ corresponds to the left action of this generating set on its Cayley graph. This is the action such that an automorphism g takes a vertex v to gv, and takes an edge connecting v to va to an edge connecting gv and gva for each generator a of $\mathrm{Aut}(F_n)$. Elementary Nielsen transformations form the same generating set, but with the action on $\mathbf{X}$ being

an interpretation of the right action of the generating set on the vertices of its Cayley graph. When restricted to the action of a generator a of $\mathrm{Aut}(F_n)$, it simply moves a vertex v across the edge connecting v to va to the vertex va. However, this right action does not extend to the edges of the Cayley graph.

In his seminal paper [Nie24], Nielsen presented a method for transforming a finite generating set for a subgroup of a free group into a free basis for that subgroup using elementary Nielsen transformations. Nielsen's method is essentially a finite reduction process, at every step of which a Nielsen transformation is used to 'simplify' the finite generating set. In Lemma 4.18 we will apply this process to the bases of F_n and will use the following fact, whose proof follows from the proof of Theorem 3.1 in [MKS04].

Proposition 2.3. *For every basis $\mathbf{x}$ of a free group F_n there is a sequence of elementary Nielsen transformations (δ_j), $1 \leq j \leq t$ taking the standard basis $\mathbf{a}$ of F_n to $\mathbf{x} = \mathbf{a}\delta_1 \ldots \delta_t$ such that the sum of the lengths (with respect to $\mathbf{a}$) of elements in the intermediate bases $\mathbf{a}\delta_1 \ldots \delta_j$ is a nondecreasing sequence.*

3. Vertices and edges in $\mathcal{ES}_n^1$

We wish to view the spaces $\mathcal{ES}_n^1$ and $\mathcal{FF}_n^1$ on which $\mathrm{Out}(F_n)$ acts in the language of ordered tuples, so that we may apply the dictionary of Section 2 equating tuples, Nielsen automorphisms, and Nielsen transformations.

We begin with an observation on elements of $\mathrm{Out}(F_n)$. An element of $\mathrm{Out}(F_n)$ is a coset of $\mathrm{Aut}(F_n)$ with respect to the subgroup $\mathrm{Inn}(F_n)$. As such, an element of $\mathrm{Out}(F_n)$ may be represented by many different n-tuples. In general, we think of an element of $\mathrm{Out}(F_n)$ as a tuple up to conjugation.

Now consider the graphs $\mathcal{ES}_n^1$ and $\mathcal{FF}_n^1$. These graphs have the same vertex set: vertices correspond to free factorizations of F_n into two nontrivial factors up to conjugation. We wish to interpret an arbitrary free factorization of F_n into two factors as a tuple, together with an index set, up to certain equivalences. Let $\mathcal{S}$ denote the set of all proper nonempty subsets of $\{1, \ldots, n\}$. We will call an element of $\mathcal{S}$ an *index set.* Then a tuple $\mathbf{x} = (x_1, \ldots, x_n)$ together with some index set $S \in \mathcal{S}$ yields a free factorization of F_n as $\langle \mathbf{x}_S \rangle * \langle \mathbf{x}_{\overline{S}} \rangle$, where $\mathbf{x}_S := \{a_i \in \mathbf{x} | i \in S\}$ and $\overline{S} := \{1, \ldots, n\} - S$. Every free factorization may be represented as a tuple/index set pair, but a given free factorization may be represented by multiple tuple/index set pairs: any tuple/index set pairs which differ by a self-map of $\mathrm{Aut}(F_n) \times \mathcal{S}$ preserving the associated free factorization up to conjugation should be identified.

Every such map can be written as a composition of four types of self-maps, defined by their action on $(\mathbf{x}, S) \in \mathrm{Aut}(F_n) \times \mathcal{S}$ as follows:

(1) conjugation of $\mathbf{x}$ without changing S,
(2) permutation of $\{1, \ldots, n\}$ applied to both $\mathbf{x}$ and S,
(3) exchanging S for $\overline{S}$ and leaving $\mathbf{x}$ unchanged,
(4) applying transformation ϕ of $\mathrm{Aut}(F_n)$ fixing the free factors in the factorization setwise (i.e. $\langle \mathbf{x}_S \rangle = \langle (\mathbf{x}\phi)_S \rangle$ and $\langle \mathbf{x}_{\overline{S}} \rangle = \langle (\mathbf{x}\phi)_{\overline{S}} \rangle$) without changing S.

The transformation ϕ in the last item is called an *S-transformation.* If there exists $S \in \mathcal{S}$ such that ϕ is an S-transformation, we call ϕ an $\mathcal{S}$-transformation.

Note that any self-map from the group mentioned above may be realized as composition of the form $m_1 m_2 m_3 m_4$, where m_i is a self-map of type (i).

With this interpretation of vertices of $\mathcal{ES}_n^1$, consider edges of $\mathcal{ES}_n^1$. Two vertices represented by $A_1 * B_1$ and $A_2 * B_2$ of $\mathcal{ES}_n^1$ are adjacent if there exists a common refinement of conjugates of the free factorizations. Such a common refinement is of the form $A * C * B$, where for some elements g and h of F_n we have $A_1^g = A * C$, $B_1^g = B$, $A_2^h = A$, and $B_2^h = C * B$. Without loss of generality we can assume that h is trivial. If $(\mathbf{x}, S)$ is the vertex corresponding to the free factorization $A_1 * B_1$, then $(\mathbf{x}^g, S)$ represents the same vertex of $\mathcal{ES}_n^1$ and the refinement $A_1^g = A * C$ corresponds to applying to $\mathbf{x}^g$ a transformation ϕ of F_n taking $\mathbf{x}_S^g$ to a basis for A union a basis for C and preserving B_1^g. Note ϕ fixes both $\langle \mathbf{x}_S^g \rangle$ and $\langle \mathbf{x}_{\overline{S}}^g \rangle$, and so is an S-transformation. Then changing $(A * C) * B$ to $A * (C * B)$ simply corresponds to subtracting from S the indices of elements in $\phi(\mathbf{x}^g)$ corresponding

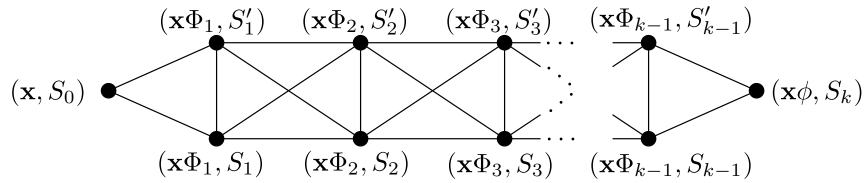

FIGURE 1. Shown are two edge paths from the vertex $(\mathbf{x}, S_0)$ to the vertex $(\mathbf{x}\phi, S_k)$ in $\mathcal{ES}_n^1$ using the notation of this section. Here, we let $\Phi_i := \phi_0 \dots \phi_{i-1}$ denote the composition of $\mathcal{S}$-transformations, where ϕ_i is an S_{i-1}-transformation. Thus, $\mathbf{x}(i) = \mathbf{x}\Phi_i = \mathbf{x}\phi_0 \dots \phi_{i-1}$. The lower path represents the edge path described in the text, and is represented by the sequence of transformations $\phi_0, \dots, \phi_k$. The upper path represents an edge path reconstructed from the $\mathcal{S}$-transformations $\phi_0, \dots, \phi_k$ by, for each $i = 1, \dots, k-1$, choosing an arbitrary index set S_i' compatible with S_{i-1}' such that ϕ_i is an S_i'-transformation. Horizontal edges in the figure are edges in $\mathcal{ES}_n^1$, and vertical and diagonal edges mean that the distance between two vertices in $\mathcal{ES}_n^1$ is at most 2.

to a basis for C. Of course, by exchanging S for $\overline{S}$, we could have instead added elements to S, which corresponds to subtracting elements from $\bar{S}$. Thus, changing $(A * C) * B$ to $A * (C * B)$ corresponds to replacing S with a proper subset of either S or $\bar{S}$. We call index sets S and S' from $\mathcal{S}$ *compatible* if either S' or $\overline{S'}$ is a proper subset of either S or $\bar{S}$.

Thus, up to conjugation, all edges from the vertex corresponding to $(\mathbf{x}, S)$ are precisely characterized by a transformation fixing $\langle \mathbf{x}_S \rangle$ and $\langle \mathbf{x}_{\overline{S}} \rangle$, followed by replacing S with a compatible element of $\mathcal{S}$. We have shown:

Lemma 3.1. *The set of edges in $\mathcal{ES}_n^1$ from a vertex $(\mathbf{x}, S)$ is determined by: a conjugation of $\mathbf{x}$, an S-transformation, and a choice of new index set compatible with S.*

An edge path p from the vertex represented by $(\mathbf{x}, S_0)$ in $\mathcal{ES}_n^1$ is described by a sequence: a conjugation γ_0 of $\mathbf{x}$, an S_0-transformation ϕ_0, a change of index set to S_1, a conjugation γ_1 of $\mathbf{x}^{\gamma_0}\phi_0$, an S_1-transformation ϕ_1, a change of index set to S_2, etc.

On such an edge path p, for any i, let $(\mathbf{x}(i), S_i)$ be a representative of the vertex on the edge path immediately before ϕ_i is to be applied. Then by construction we have

$$\mathbf{x}(i) = \big(\dots\big((\mathbf{x}^{\gamma_0}\phi_0)^{\gamma_1}\phi_1\big)^{\gamma_2}\dots\big)^{\gamma_{i-1}}\phi_{i-1} = (\mathbf{x}\phi_0\phi_1\dots\phi_{i-1})^{\left(\dots\left((\gamma_0)\phi_0\gamma_1\right)\phi_1\gamma_2\dots\right)\phi_{i-1}}.$$

As vertices of $\mathcal{ES}_n^1$ are only defined up to conjugation, we may assume without loss of generality that all of the conjugators γ_i are trivial and

$$\mathbf{x}(i) = \mathbf{x}\phi_0\phi_1\dots\phi_{i-1}.$$

The set S_i is not determined uniquely by ϕ_i, as ϕ_i may be an S-transformation for many index sets S. However, for any such S, the vertex $(\mathbf{x}(i), S)$ is of distance at most 2 away from each of the vertices $(\mathbf{x}(i-1), S_{i-1})$, $(\mathbf{x}(i), S_i)$, and $(\mathbf{x}(i+1), S_{i+1})$ in $\mathcal{ES}_n^1$, as follows. That $(\mathbf{x}(i), S)$ is distance at most 2 from $(\mathbf{x}(i-1), S_{i-1})$ follows from applying ϕ_{i-1} to $(\mathbf{x}(i-1), S_{i-1})$ and then changing the index set to S, which requires one edge in $\mathcal{ES}_n^1$ if S_{i-1} and S are compatible and 2 edges otherwise. That $(\mathbf{x}, S)$ is distance at most 2 from $(\mathbf{x}(i), S_i)$ follows from applying the identity transformation to $(\mathbf{x}(i), S_i)$ (note the identity transformation is indeed an S_i-transformation) and then changing index set to S_i. Finally, for the vertex $(\mathbf{x}(i+1), S_{i+1})$, since ϕ_i is an S-transformation, $(\mathbf{x}(i+1), S_{i+1})$ is the vertex obtained from $(\mathbf{x}, S)$ by applying ϕ_i to $\mathbf{x}(i)$ and then changing the index set to S_{i+1}.

Thus, up to distance 2 at every vertex on the path p, the path p is determined by the sequence of transformations $\phi_0, \phi_1, \dots, \phi_k$ (see Figure 1). Note that we may reverse this procedure: take a sequence of transformations $\phi_0, \phi_1, \dots, \phi_k$ such that each ϕ_i is an $\mathcal{S}$-transformation, choose any $S_i' \in \mathcal{S}$ such that ϕ_i is an S_i'-transformation, and obtain an edge path in $\mathcal{ES}_n^1$, which is uniquely defined up to distance 2 at each vertex.

A geodesic in $\mathcal{ES}_n^1$ is then easy to describe. A geodesic, up to distance 2 at each vertex, is an edge path $\phi_0, \phi_1, \ldots, \phi_k$ such that the transformation $\phi = \phi_0\phi_1 \ldots \phi_k$ is not a product of fewer than $k+1$ $\mathcal{S}$-transformations with the property that the neighboring transformations are $\mathcal{S}$-transformations with respect to compatible index sets.

For a given word w in the generating set for $\mathrm{Aut}(F_n)$ consisting of elementary Nielsen transformations and the identity transformation, we say that w has *at most k index changes* if w may be expressed as a product of $k+1$ disjoint subwords, each of which is an $\mathcal{S}$-transformation and the neighboring subwords are $\mathcal{S}$-transformations with respect to compatible index sets. If k is minimal over all such products, we say *w requires k index changes*. Since the product of S-transformations is an S-transformation, we can rephrase the preceding paragraph in the form of the following Theorem.

Theorem 3.2. *A geodesic in $\mathcal{ES}_n^1$ is represented by a product of $\mathcal{S}$-transformations with the minimal number of index changes. Moreover, a geodesic of length k in $\mathcal{ES}_n^1$ requires from $k-4$, to k index changes.*

We will use this characterization to describe lower bounds on distances in $\mathcal{ES}_n^1$ based on properties of the associated transformations in Section 4.

We end this section by noting that there is a similar characterization of roses in the spine K_n of outer space as tuples, up to conjugation and signed permutation (the signed permutations correspond to graph isomorphisms). With this interpretation, there are canonical Lipschitz maps from K_n to $\mathcal{ES}_n^1$ to $\mathcal{FF}_n^1$. It is also worth noting that the quasiisometry between $\mathrm{Out}(F_n)$ and K_n may be stated in this language: Let K_n' be the graph whose vertices are the marked roses of K_n and whose edges correspond to marked roses lying on a common 2-cell in K_n. Then K_n is 2-biLipschitz equivalent to the graph K_n', and K_n' is biLipschitz equivalent to the Cayley graph of $\mathrm{Out}(F_n)$ with respect to the generating set of elementary Whitehead transformations: K_n' is the Schreier graph of $\mathrm{Out}(F_n)$ with respect to this generating set and the finite subgroup of signed permutations.

4. The Notion of i-Length

In this section, we define the notion of *i-length* and analyze its properties. This notion is an algebraic tool that will be used to estimate distances in $\mathcal{ES}_n^1$. We use the concept of i-length to refer to a measure of complexity of 3 different kinds of objects: abstract words in the generators of F_n, subwords of bases of F_n, and bases of F_n themselves. Our 3 concepts of i-length are: *simple* i-length, *conjugate reduced* i-length, and *full* i-length, respectively. We use simple i-length to define conjugate reduced i-length, and conjugate reduced i-length to define full i-length. After defining the three notions of i-length, we will analyze the properties of each in turn.

Throughout this section, we fix a standard basis $\mathbf{a} = \{a_1, \ldots, a_n\}$ of F_n once and for all.

4.1. Defining i-Length.

We motivate our definition of i-length with an example.

Let $H := \langle a_1, \ldots, a_{n-1}\rangle$ denote the subgroup of F_n of rank $n-1$ corresponding to ignoring the generator a_n. Consider the vertex $v_0 := [H * \langle a_n\rangle]$ as a basepoint in $\mathcal{ES}_n^1$, and think about moving in $\mathcal{ES}_n^1$ to the vertex $v = [H * \langle a_n h\rangle]$, where h is an arbitrary element of H. Let d denote the distance between v_0 and v in $\mathcal{ES}_n^1$.

If h is nontrivial, then $v \neq v_0$, as there is clearly no way of using conjugation to remove occurrences of all elements of H from the second factor of any representative of v. Moreover, as v_0 and v both have the same index set, by Theorem 3.2, when h is nontrivial we have $d \geq 2$.

If h is a primitive element in H, then $d = 2$, as follows. Let $h_2, \ldots, h_{n-1}$ denote elements of H such that $\{h, h_2, \ldots, h_{n-1}\}$ forms a basis for H. Then $\langle h, h_2, \ldots, h_{n-1}\rangle * \langle a_n\rangle$ is a representative of v_0, and $\langle h, h_2, \ldots, h_{n-1}\rangle * \langle a_n h\rangle$ is a representative of v. Thus, $[\langle h_2, \ldots, h_{n-1}\rangle * \langle h, a_n\rangle]$ is a vertex which is adjacent to both v and v_0.

If h is a power of a primitive element in H, the same argument again shows that $d = 2$. Figure 2 shows the path of length 2 connecting $[H * \langle a_n\rangle]$ and $[H * \langle a_n h\rangle]$ for $n = 3$, $h = g^k$, where g is primitive in H and g_2 is some coprimitive with g element such that $\langle g, g_2\rangle = H$. Repeating the above argument shows that, if we know that h is a product of j powers of primitive elements in

$$\begin{tikzcd}
{[\langle g, g_2\rangle * \langle a_3\rangle]} \arrow[r, dotted] \arrow[d, equal] & {[\langle g_2\rangle * \langle a_3, g\rangle]} \arrow[d, equal] & {[H * \langle a_3 h\rangle]} \arrow[d, equal] \\
{[H * \langle a_3\rangle]} & {[\langle g_2\rangle * \langle a_3 g^k, g\rangle]} \arrow[r, dotted] & {[\langle g, g_2\rangle * \langle a_3 g^k\rangle]}
\end{tikzcd}$$

FIGURE 2. A path of length 2 in $\mathcal{ES}_3^1$.

H, then $d \leq 2j$. To obtain a lower bound on d, we need to at least minimize j. Thus, we need to consider how to detect how many powers of primitives are needed to form h.

One property of a (power of a) primitive element h of H is a classical result of Whitehead, which states that the *Whitehead graph* of h, considered as a reduced word in the alphabet $(\mathbf{a} - \{a_n\})^{\pm 1}$, must have a *cut vertex*, defined as follows.

Definition 4.1 (Whitehead graph)**.** For a set of freely reduced words $\mathbf{x} = \{x_1, \dots, x_k\}$ in the alphabet $\mathbf{a} \cup \mathbf{a}^{-1}$, define the *Whitehead graph* $\Gamma_{\mathbf{a}}(\mathbf{x})$ as follows. The set of vertices of $\Gamma_{\mathbf{a}}(\mathbf{x})$ is identified with the set $\mathbf{a} \cup \mathbf{a}^{-1}$. For every $x_i \in \mathbf{x}$ of length n, x_i contributes exactly $n-1$ edges to $\Gamma_{\mathbf{a}}(\mathbf{x})$, one for each pair of consecutive letters in x_i. The edge added for a $a_i a_j$ is from the vertex a_i to the vertex a_j^{-1}. The *augmented Whitehead graph* $\hat{\Gamma}_{\mathbf{a}}(\mathbf{x})$ is the Whitehead graph $\Gamma_{\mathbf{a}}(\mathbf{x})$ together with an additional edge for each $x_i \in \mathbf{x}$, from the last letter of x_i to the inverse of the first letter. In particular, a word $x_i = a_j$ of length 1 contributes exactly one edge, from a_j to a_j^{-1}, to $\hat{\Gamma}_{\mathbf{a}}(\mathbf{x})$. For a single word w, we abuse notation and write $\Gamma_{\mathbf{a}}(w)$ for $\Gamma_{\mathbf{a}}(\{w\})$ and $\hat{\Gamma}_{\mathbf{a}}(w)$ for $\hat{\Gamma}_{\mathbf{a}}(\{w\})$.

If $\mathbf{x}$ is cyclically reduced and linearly independent, then the Whitehead graph of a set of freely reduced words $\mathbf{x}$ is graph-isomorphic to the link of the unique vertex in the presentation 2-complex of the group $F_n/\langle\langle \mathbf{x}\rangle\rangle$ generated by $a_1, \dots, a_n$ with relations $x_1, \dots, x_k$.

Note that a Whitehead graph (or augmented Whitehead graph) may have multiple edges. Loops at a vertex may appear only in an augmented Whitehead graph and if and only if at least one of the words in $\mathbf{x}$ is not cyclically reduced. An example of the augmented Whitehead graph, namely $\hat{\Gamma}_{\{a_1,a_2,a_3,a_4\}}(a_2^2 a_3^2 a_4^2)$, is shown in Figure 3.

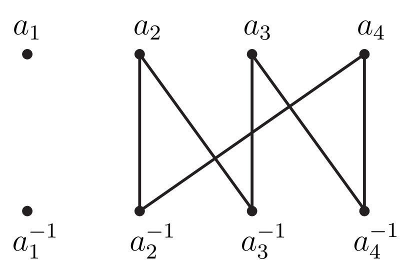

FIGURE 3. Augmented Whitehead graph $\hat{\Gamma}_{\{a_1,a_2,a_3,a_4\}}(a_2^2 a_3^2 a_4^2)$

Definition 4.2 (cut vertex)**.** A *cut vertex* v of a graph Γ is a vertex such that $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are nonempty subgraphs and $\Gamma_1 \cap \Gamma_2 = \{v\}$. If Γ is disconnected, then all of its vertices are cut vertices.

Whitehead proved [Whi36] that the augmented Whitehead graph of a basis of a free group has a cut vertex. Note that a power of a primitive has the same augmented Whitehead graph as the given primitive, so the augmented Whitehead graph of a power of a primitive must also have a cut vertex. The converse is, of course, not true – for example, aba^3b is not primitive in F_2 – but of course the contrapositive is: having an augmented Whitehead graph with no cut vertex implies the element is not a primitive or a power of a primitive.

For our purposes, we will need a generalization of Whitehead's theorem due to Stallings [Sta99], so we state it now. A subset S of F_n is called *separable* if there is a free factorization of F_n with

two factors such that each element of S can be conjugated into one of the factors. In particular, a set is separable if its elements can be conjugated (possibly by different conjugators) to the elements of some basis of F_n. Thus, a basis (and the cyclic reduction of a basis) is always separable.

Theorem 4.3 ([Sta99])**.** *If $\mathbf{x}$ is a separable set in F_n, then there is a cut vertex in $\hat{\Gamma}_{\mathbf{a}}(\mathbf{x})$.*

Now consider our motivating example of the distance d between $v_0 = [H * \langle a_n \rangle]$ and $v = [H * \langle a_n h \rangle]$ in $\mathcal{ES}^1_n$. Naïvely, we could hope that if we could break up h, considered as a reduced word, into k subwords such that each subword had an augmented Whitehead graph with no cut vertex, then d might be bounded from below by a function of k. However, it may not be the case that such a decomposition of h 'breaks' h in the places corresponding to the most efficient way of decomposing it as a product of powers of primitives: a given primitive might contribute to one or more of the subwords. But Whitehead's theorem does *not* say that the (augmented) Whitehead graph of any subword of a primitive will have a cut vertex. Indeed, a primitive element conjugated by an arbitrary word will still be primitive, and the only reason its augmented Whitehead graph will have a cut vertex will be from the single self-loop contributed by the last and first letters. If the primitive element is cyclically reduced, then we may claim that the (non-augmented or augmented) Whitehead graph of any subword will have a cut vertex, but not otherwise.

The notions of i-length are defined precisely to deal with this delicate effect of conjugation. Simple i-length ignores conjugation completely, looking only at the non-augmented Whitehead graph of a word and its subwords. Conjugate reduced i-length takes all possible conjugations of the subwords of a word into account. Full i-length then uses conjugate reduced i-length to measure the complexity of an entire basis.

We are almost ready to give the definitions of i-length, but we need one minor piece of notation to proceed.

Notation 4.4. Elements of F_n are equivalence classes of words in the alphabet $\mathbf{a} \cup \mathbf{a}^{-1}$ under free reduction. For two words w_1 and w_2 in this alphabet, we write $w_1 = w_2$ if they are equal as words, and $w_1 =_r w_2$ if they are equal after free reduction, i.e. as elements of F_n.

We now define the 3 notions of i-length. The definition of full i-length is somewhat nuanced, but the concept is based on simple i-length. The idea of simple i-length is straightforward: it records the maximal number of pieces a word can be broken into such that the Whitehead graph of each piece has no cut vertex.

Definition 4.5 (Simple i-length)**.** Fix an index $i \in \{1, \dots, n\}$. Let w be a word which contains no occurrence of $a_i^{\pm 1}$. The *simple i-length* of w, denoted $|w|_i^{simple}$, is the greatest number t such that w is of the form $w_1 w_2 \dots w_t$, where $\Gamma_{\mathbf{a}-\{a_i\}}(w_j)$ has no cut vertex for each $j = 1, \dots, t$. If $\Gamma_{\mathbf{a}-\{a_i\}}(w)$ has a cut vertex, we define $|w|_i^{simple}$ to be zero.

It worth pointing out that in the above definition we use standard Whitehead graph, not the augmented one.

Conjugate reduced i-length additionally takes conjugation into account.

Definition 4.6 (Conjugate reduced i-length)**.** Fix an index $i \in \{1, \dots, n\}$. Let w be a word which contains no occurrence of $a_i^{\pm 1}$ (thought of as a subword of another word in the alphabet $\mathbf{a}^{\pm 1}$). Then w has *conjugate reduced i-length at most k* if there exist freely reduced words $v_1, \dots, v_l, u_1, \dots, u_l$ such that:

(1) $w =_r v_1^{u_1} v_2^{u_2} \dots v_l^{u_l}$, where $v_j^{u_j} := u_j^{-1} v_j u_j$, and
(2) $k = (l-1) + |v_1|_i^{simple} + \dots + |v_l|_i^{simple}$.

The decomposition of w as $v_1^{u_1} v_2^{u_2} \dots v_l^{u_l}$ is called a *decomposition*, and k is the *conjugate reduced i-length associated to the decomposition*. If the associated k is minimal among all such decompositions, the decomposition is called *optimal*, and k is called a conjugate reduced i-length of w and denoted by $|w|_i^{cr}$. The number l of factors of the form $v_j^{u_j}$ in the decomposition is called the *factor length* of the decomposition.

We are now ready to define the (full) i-length of a basis for F_n (or more generally a set of words). Given a basis Y, we essentially measure the maximal conjugate reduced i-length of any subword

of any element of Y. However, we must be very careful to properly account for conjugation. We do so as follows.

Let $\mathbf{y}$ be a set of reduced words in the alphabet $\mathbf{a}^{\pm 1}$. Let $\tilde{\mathbf{y}}$ denote the set of elements of $\mathbf{y}$ after each of them has been cyclically reduced. Define $w_L = w_L(\mathbf{y})$ to be the longest word in the alphabet $(\mathbf{a} - \{a_i\})^{\pm 1}$ such that every occurrence of a_i in every $\tilde{y} \in \tilde{\mathbf{y}}$ is cyclically preceded by w_L and every occurrence of a_i^{-1} is cyclically followed by w_L^{-1} (note w_L could be trivial). Similarly, let $w_R = w_R(\mathbf{y})$ be the longest word in $(\mathbf{a} - \{a_i\})^{\pm 1}$ such that every occurrence of a_i in every $\tilde{y} \in \tilde{\mathbf{y}}$ is cyclically followed by w_R, every occurrence of a_i^{-1} is cyclically preceded by w_R^{-1}, and no such occurrence of w_R intersects any such occurrence of w_L (again, w_R could be trivial). Let $\alpha' = \alpha'_{\mathbf{y}}$ be the automorphism of F_n which maps a_i to $w_L^{-1} a_i w_R^{-1}$. Let $w_C = w_C(\mathbf{y})$ be the longest word in $(\mathbf{a} - \{a_i\})^{\pm 1}$ such that, in $\alpha'\tilde{\mathbf{y}}$, every occurrence of a_i^k either: (a) occurs by itself as an element of $\alpha'\tilde{\mathbf{y}}$ or (b) appears cyclically conjugated by w_C, so that a_i^k is cyclically preceded by w_C^{-1} and cyclically followed by w_C. If every occurrence of a_i^k occurs by itself, we declare that w_C is trivial.

If $\mathbf{y}$ is a singleton $\{y\}$, we abuse notation and write y instead of $\{y\}$ when applying any function in this subsection.

Let $\alpha = \alpha_{\mathbf{y}}$ be the automorphism of F_n which maps a_i to $w_L^{-1} w_C a_i w_C^{-1} w_R^{-1}$. Thus, $w_L(\alpha\mathbf{y}) = w_R(\alpha\mathbf{y}) = 1$ are trivial. The preimage of a_i^k under α, after free reduction, is $w_C^{-1}\ (w_L a_i w_R)^k\ w_C$. Note that w_C^{-1} and w_C canceled between adjacent occurrences of a_i, but that this is the only free cancelation which occurs in $\alpha^{-1}(\alpha\mathbf{y})$. Note $\alpha\mathbf{y}$ may not be cyclically reduced.

An *i-chunk* of a word y in the alphabet $\mathbf{a}^{\pm 1}$ is a cyclic subword of $\tilde{y}$ (here again, $\tilde{y}$ denotes the result of a cyclic reduction of y) which contains no $a_i^{\pm 1}$ and is maximal among such subwords ordered by inclusion. By definition, every i-chunk of y begins with either $w_R(y)$ or $(w_L(y))^{-1}$, and ends with either $w_L(y)$ or $(w_R(y))^{-1}$.

For example, in the set $\mathbf{y} = \{a_2^{-1} a_3 a_4 a_1 a_4 a_2 a_3,\ a_4^{-1} a_1^{-1} a_4^{-1} a_3^{-1}\}$, we have $\tilde{\mathbf{y}} = \mathbf{y}$. For $i = 1$, $w_L(\mathbf{y}) = a_3 a_4$, $w_R(\mathbf{y}) = a_4$, and $w_C(\mathbf{y}) = a_2$. Thus, $\alpha(a_1) = a_4^{-1} a_3^{-1} a_2 a_1 a_2^{-1} a_4^{-1}$, so that $\alpha(\mathbf{y}) = \{a_1 a_3,\ a_2 a_1^{-1} a_2^{-1}\}$.

Definition 4.7 (Full i-length)**.** Fix an index $i \in \{1, \dots, n\}$. Let $\mathbf{y}$ be a set of words in the alphabet $\mathbf{a}^{\pm 1}$. The (full) *i-length* of $\mathbf{y}$ is

$$|\mathbf{y}|_i := k(\mathbf{y}) + |w_R(\mathbf{y}) w_L(\mathbf{y})|_i^{cr},$$

where $k(\mathbf{y})$ is the maximal conjugate reduced i-length of an i-chunk of $\alpha_{\mathbf{y}} y$ over all elements $y \in \mathbf{y}$.

For example, let $w = a_1^2 a_2^2 \dots a_{n-1}^2 a_1$. Then $|w|_n^{simple} = |w|_n^{cr} = 1$, and $|a_n w|_n = 1$. We will later see (in Corollary 4.14) that $|a_n w^l|_n \geq l/3 - 2$.

4.2. **Properties of Simple i-length.**

This subsection includes three simple lemmas that we will use in further proofs.

Lemma 4.8. *Let w be a freely reduced word in F_n which contains no occurrence of $a_i^{\pm 1}$ and let u and v be disjoint subwords of w. Then*

$$|w|_i^{simple} \geq |u|_i^{simple} + |v|_i^{simple}.$$

Proof. If $|u|_i^{simple} > 0$ and $|v|_i^{simple} > 0$ then consider the partitions of u and v into $|u|_i^{simple}$ and $|v|_i^{simple}$ pieces respectively. These partitions induce a partition of w into $|u|_i^{simple} + |v|_i^{simple}$ pieces, where the portions of w disjoint from u and v are appended to the first and last pieces in the partitions of u and v. Note that appending will not brake the fact that the Whitehead graph of a piece does not have a cut vertex, because w is freely reduced. If $|u|_i^{simple} = 0$ (respectively, $|v|_i^{simple} = 0$) then we similarly form a partition of w into $|v|_i^{simple}$ (respectively, $|u|_i^{simple}$) pieces. In the case $|u|_i^{simple} = 0$ and $|v|_i^{simple} = 0$ the claim is trivial. □

Lemma 4.9. *Let u and v be freely reduced words which contain no occurrence of $a_i^{\pm 1}$ such that $w = uv$ is freely reduced. Then*

$$|w|_i^{simple} \leq |u|_i^{simple} + |v|_i^{simple} + 1.$$

Proof. If $|w|_i^{simple} = 0$ then the claim is trivial. Otherwise let $w = w_1w_2\cdots w_k$ be a partition of w realizing the simple i-length of w, and let j denote the first index such that w_j is not fully contained in u. This gives partitions $u = w_1w_2\cdots(w_{j-1}w'_j)$ and $v = (w''_jw_{j+1})w_{j+2}\cdots w_k$ of u and v showing that

$$|u|_i^{simple} + |v|_i^{simple} \geq (j-1) + (k-j) = k-1 = |w|_i^{simple} - 1.$$

□

Lemma 4.10. *If w is a cyclically reduced word which contains no occurrence of $a_i^{\pm 1}$ and w' is a cyclic conjugate of w, then*

$$|w|_i^{simple} - 1 \leq |w'|_i^{simple} \leq |w|_i^{simple} + 1.$$

Proof. It is enough to show that cyclic conjugation cannot decrease the simple i-length by more than 1. Let $\iota(w)$ be the initial segment of w such that $w' =_r w^{\iota(w)}$. Let $w = w_1w_2\cdots w_k$ be the partition of w realizing the simple i-length of w, and let j denote the first index such that w_j is not fully contained in $\iota(w)$. Then w' can be partitioned as $(w''_jw_{j+1})\ldots w_kw_1\ldots(w_{j-1}w'_j)$ where $w_j = w'_jw''_j$. Thus, w' can be partitioned into at least $k-1$ subwords of nontrivial simple i-length, and the lemma follows. □

4.3. **Properties of Conjugate reduced i-Length.**

We now wish to describe some properties of conjugate reduced i-length. However, before we do so, we need to verify that conjugate reduced i-length is not a trivial notion of complexity. In this section, we show that there exist words of arbitrary conjugate reduced i-length. In the process, we develop a useful lemma for working with i-length. Then, we collect three short lemmas which describe how conjugate reduced i-length is related to simple i-length, and how conjugate reduced i-length behaves under multiplication.

Definition 4.11 (canceling pairs)**.** Let $w \in F_n$ be arbitrary reduced word which contains no occurrence of $a_i^{\pm 1}$. A set of any two subwords of w of the form u, u^{-1} is called a *canceling pair in w*. A family $\mathcal{F}$ of canceling pairs in w is called *nested* if canceling pairs in $\mathcal{F}$ are disjoint and, for any canceling pairs u, u^{-1} and v, v^{-1} in $\mathcal{F}$, v occurs between u and u^{-1} in w if and only if v^{-1} does. If $\mathcal{F}$ is a nested family of canceling pairs for w, we abuse notation and let $w - \mathcal{F}$ denote the set of subwords of w which are maximal under inclusion and which do not intersect any element of any canceling pair in $\mathcal{F}$ as subwords of w. Finally, we define $|w-\mathcal{F}|_i^{simple} := |\mathcal{F}| + \sum_{w' \in (w-\mathcal{F})} |w'|_i^{simple}$.

Lemma 4.12. *Let $w \in F_n$ be a nontrivial reduced word which contains no occurrence of $a_i^{\pm 1}$ and let T be the set consisting of all nested families of canceling pairs of w. Then*

$$|w|_i^{cr} \geq \min_{\mathcal{F} \in T} \left(\max\left\{ \frac{|\mathcal{F}|}{2} - 1,\ \frac{1}{5}|w - \mathcal{F}|_i^{simple} - 3 \right\} \right).$$

Proof. Let $\gamma = v_1^{u_1}v_2^{u_2}\cdots v_l^{u_l}$ be an optimal decomposition of w realizing its conjugate reduced i-length.

First of all, since γ is optimal, we may assume that all v_j are cyclically reduced (cyclic reduction of v_j cannot increase the conjugate reduced i-length).

Now we will utilize the technique used in the proof of the van Kampen lemma (see, for example, [LS01]). The word w represents a trivial element in the group defined by the presentation

$$\langle \mathbf{a} \mid v_1, v_2, \ldots, v_l \rangle. \tag{1}$$

Consider the van Kampen diagram Γ_0 with boundary label γ over the presentation (1) as depicted in Figure 4. This diagram is a wedge of l “lollipops” corresponding to l factors of γ with “stems” labeled by the u_j and with the “candies” (boundaries of 2-cells) labelled by the v_j. The base-vertex in Γ_0 is the common vertex of “lollipops”.

Fix some free reduction process transforming γ to w. The jth step of this reduction process takes the van Kampen diagram Γ_{j-1} to the diagram Γ_j, and corresponds to modifying a pair of adjacent, inversely labeled edges along the boundary cycle of Γ_{j-1}. This has the effect of ‘removing’ this pair of edges from the boundary cycle of Γ_j in the following sense. If these two edges have just one vertex in common, they are folded and if this common vertex has degree 2 in Γ_{j-1} then the

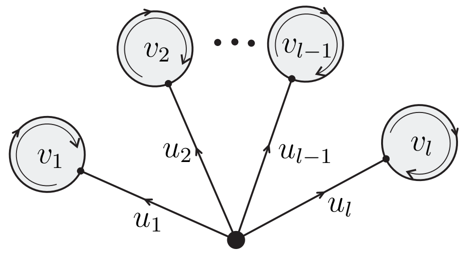

FIGURE 4. van Kampen diagram Γ_0 corresponding to decomposition γ

edge obtained by folding is removed from Γ_{j-1}. If they have two vertices in common, the union of 2-cells bounded by these two edges is completely removed from Γ_{j-1}. This folding or removing defines a new van Kampen diagram Γ_j. At the end of the process we obtain a van Kampen diagram Γ with boundary label w shown in Figure 5. Note that in the reduction process the number of 2-cells in each successive van Kampen diagram does not grow, so the number l' of 2-cells in Γ does not exceed l.

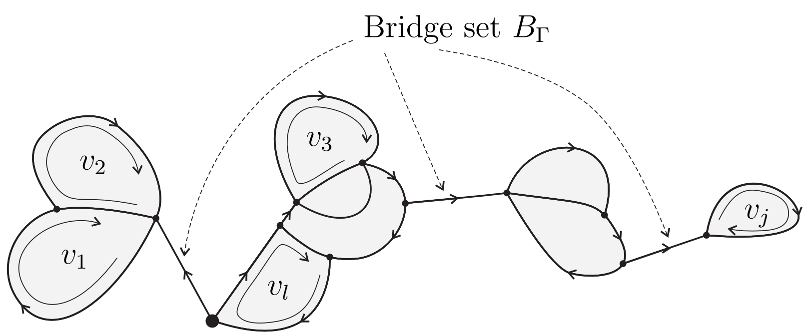

FIGURE 5. van Kampen diagram Γ after folding

Because each of the v_j's is cyclically reduced, the boundary of each 2-cell in the diagram Γ is labeled by a cyclic conjugate of v_j that depends on where along the boundary one begins reading. The *bridge set* B_Γ of Γ is the set of all vertices and edges whose deletion from the topological realization $|\Gamma|$ of Γ would disconnect it. A *disk-component* of Γ is a subset of Γ which is the closure of a connected component of $|\Gamma| - |B_\Gamma|$. The disk-components of Γ are joined by (possibly trivial) edge-paths from the bridge set. Retracting each of these paths to a point produces a new van Kampen diagram Γ' with a boundary label u obtained from w by removing a nested family of canceling pairs, denoted $\mathcal{F}$, where each canceling pair corresponds to a path inside B_Γ whose inner vertices have degree 2. Such a diagram is depicted in Figure 6. Note that u is not necessarily freely reduced, but that u is the product of subwords in $w - \mathcal{F}$, all of which are subwords of w and hence freely reduced. The vertices of degree at least three along the boundary of Γ' split u into subwords $w_1, w_2, \ldots, w_k$, where each w_j is a part of the boundary of a 2-cell in Γ. This partition of u refines the partition $w'_1, w'_2, \ldots, w'_r$ of u induced by $w - \mathcal{F}$.

Collapsing all disc components of Γ and removing vertices of degree 2 leaves the tree with e edges and r' vertices of degree 1, each of which was obtained by collapsing one of the disc components. In every such tree we have $e \leq 2r'$. For the number of canceling pairs in $\mathcal{F}$ we get $|\mathcal{F}| = e + r''$, where r'' is the number of disc components in Γ that collapse to the vertices of degree 2. But since each disc component produces at least one w'_j, we get

$$|\mathcal{F}| = e + r'' \leq 2r' + r'' \leq 2(r' + r'') \leq 2r \tag{2}$$

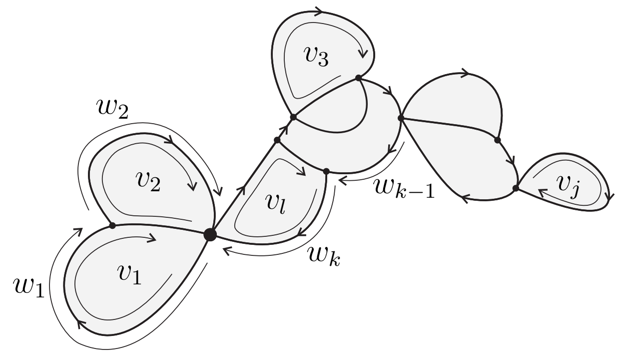

FIGURE 6. van Kampen diagram Γ' after bridge retraction

We have by Lemma 4.9 and the last inequality that

$$|w-\mathcal{F}|_i^{simple} = |\mathcal{F}| + \sum_{j=1}^{r} |w'_j|_i^{simple} \leq 2r + \sum_{j=1}^{k} |w_j|_i^{simple} + (k-r) = k + r + \sum_{j=1}^{k} |w_j|_i^{simple}. \tag{3}$$

By construction each w_j is a subword of a cyclic conjugate v'_t of some v_t representing the label of the boundary of 2-cell in Γ' to which w_j belongs. It may happen that several w_j lie on the boundary of one cell labelled by a conjugate of v_t, but by construction these occurences do not overlap. Let $\{c_1, \ldots, c_l\}$ denote the set of cells in Γ' and assume that v_j is a boundary label of c_j. Then the sum in (3) can be rewritten as

$$|w-\mathcal{F}|_i^{simple} \leq k + r + \sum_{t=1}^{l} \sum_{w_j \in \partial c_t} |w_j|_i^{simple} \tag{4}$$

Since by Lemmas 4.8 and 4.10,

$$\sum_{w_j \in \partial c_t} |w_j|_i^{simple} \leq |v'_t|_i^{simple} \leq |v_t|_i^{simple} + 1,$$

we can transform inequality (4) to

$$|w-\mathcal{F}|_i^{simple} \leq k + r + \sum_{t=1}^{l} (|v_t|_i^{simple} + 1) = |w|_i^{cr} + k + r + 1 \leq |w|_i^{cr} + 2k + 1. \tag{5}$$

To finish the proof of the lemma we first prove that $k \leq 2l' - 1 \leq 2l - 1$. This follows by induction on the number l' of cells of Γ as follows. Clearly if $l' = 1$ then $k = 1$. Assume that for any bridge-free van Kampen diagram with $l' - 1$ 2-cells the number of arcs along the boundary without vertices of degree at least 3 is at most $2(l'-1) - 1$. Choose a 2-cell c in Γ whose boundary contains a piece p of boundary of Γ and such that after p and interior of c are removed from Γ the resulting diagram $\Gamma - p$ is still bridge-free and connected. There are several cases describing how p may be attached to the boundary of $\Gamma - p$. It is straightforward to check that, in all cases, the attaching of p can increase the number of arcs without vertices of degree at least 3 by at most 2.

Finally, consider two cases. If $|w-\mathcal{F}|_i^{simple} \leq 5l - 5$, then

$$|w|_i^{cr} \geq l - 1 \geq \frac{1}{5} |w-\mathcal{F}|_i^{simple}.$$

But if $|w-\mathcal{F}|_i^{simple} > 5l - 5$, then since $k \leq 2l - 1 < \frac{2}{5}|w-\mathcal{F}|_i^{simple} + 1$ we get from (5)

$$|w|_i^{cr} \geq |w-\mathcal{F}|_i^{simple} - 2k - 1 > \frac{1}{5}|w-\mathcal{F}|_i^{simple} - 3.$$

This proves half of the lemma.

For the second part of the lemma note that by 2

$$|\mathcal{F}| \leq 2(r' + r'') \leq 2l$$

since $r' + r''$ does not exceed the number of all disk components in Γ and each disk component contains at least one cell. Thus,

$$|w|_i^{cr} \geq l - 1 \geq \frac{|\mathcal{F}|}{2} - 1.$$

The statement of the lemma now follows. □

Corollary 4.13. *If w is a positive word, then*

$$|w|^{cr} \geq \frac{1}{5}|w|^{simple} - 3.$$

Proof. If w is positive, then the only possible family of canceling pairs is the trivial family. □

Corollary 4.14. *There exist words of arbitrary (simple, conjugate reduced) i-length; there exist bases of F_n of arbitrary full i-length.*

Proof. It now follows from the previous corollary that, for $w = a_1^2 a_2^2 \dots a_{n-1}^2 a_1$,

$$|a_n w^l|_n = |w^l|_n^{cr} \geq l/5 - 3.$$

□

We now state a lemma relating simple and conjugate reduced i-length.

Lemma 4.15. *For any reduced word w,*

$$|w|_i^{simple} \geq |w|_i^{cr}.$$

If $|w|_i^{cr} > 0$, then the Whitehead graph $\Gamma_{\mathbf{a}-\{a_i\}}(w)$ has no cut vertex.

Proof. A word w represents a decomposition of itself with one factor whose conjugate reduced i-length is equal by definition to $|w|_i^{simple}$.

If $|w|_i^{cr} > 0$, then $|w|_i^{simple} > 0$. If the Whitehead graph $\Gamma_{\mathbf{a}-\{a_i\}}(w)$ had a cut vertex, then since w is freely reduced the Whitehead graph of any subword of w would also have a cut vertex. This contradicts that $|w|_i^{simple} > 0$. □

Finally, we have the two lemmas describing how conjugate reduced i-length behaves under multiplication.

Lemma 4.16. *For any words u and v, we have*

$$|u|_i^{cr} - |v|_i^{cr} - 1 \leq |uv|_i^{cr} \leq |u|_i^{cr} + |v|_i^{cr} + 1.$$

Proof. A decomposition of uv may be obtained by concatenating optimal decompositions of u and v. The associated i-length of this decomposition of uv yields the second inequality. The first inequality follows from the second inequality by concatenating uv and v^{-1}: $|u|_i^{cr} = |uvv^{-1}|_i^{cr} \leq |uv|_i^{cr} + |v|_i^{cr} + 1$. □

Lemma 4.17. *For any words u, v, and w, we have*

$$|uv|_i^{cr} - |w|_i^{cr} - 1 \leq |uwv|_i^{cr} \leq |uv|_i^{cr} + |w|_i^{cr} + 1.$$

Proof. We prove that $|uwv|_i^{cr} \leq |uv|_i^{cr} + |w|_i^{cr} + 1$; the first inequality will then follow by observing $|uv|_i^{cr} = |uww^{-1}v|_i^{cr} \leq |uwv|_i^{cr} + |w|_i^{cr} + 1$.

Consider an optimal decomposition of uv,

$$\omega = v_1^{u_1} \dots v_l^{u_l},$$

so that $|uv|_i^{cr} = \sum_j |v_j|_i^{simple} + l - 1$. We will alter this optimal decomposition of uv to obtain a decomposition of uwv, at the price of possibly introducing a bounded amount of additional conjugate reduced i-length. The lemma will then follow.

The decomposition ω freely reduces to uv, or put another way the word ω is obtained from uv by a sequence of words, each of which differs from the previous one by inserting a single canceling

pair of letters. Throughout this process, we may split each word into two halves, the left half and the right half, as follows. Begin by declaring u is the left half of uv and v is the right half. If a canceling pair is inserted into the middle of either half of a word, insert the canceling pair in the appropriate half to obtain the new halves. If a canceling pair $b^{-1}b$ is inserted between the left half and the right half, add b^{-1} to the left half and add b to the right half to obtain the new halves.

Let p be the smallest index such that the left half of ω is contained in $u_1^{-1}v_1u_1\dots u_p^{-1}v_pu_p$.

The split between halves of ω will either be in u_p^{-1}, v_p, u_p or between them.

If the split occurs in v_p then at the price of splitting $v_p^{u_p}$ into a product $(v_p')^{u_p}(v_p'')^{u_p}$, we may assume that the left half of ω is equal to $u_1^{-1}v_1u_1\dots u_p^{-1}v_pu_p$, making the right half of ω equal to $v_{p+1}^{u_{p+1}}\dots v_l^{u_l}$. Possibly splitting v_p could increase associated conjugate reduced i-length by 1.

If the split happens in, immediately before, or immediately after u_p then $u_p = u_p'u_p''$ where u_p' is in the left half of ω and u_p'' is in the right half of ω. Note that u_p' or u_p'' can be trivial.

Consider inserting w into ω, between u_p' and u_p''. Any optimal decomposition of w conjugated by $(u_p'')^{-1}$ inserted into ω then yields a decomposition of uwv. The associated conjugate reduced i-length is

$$|w|_i^{cr} + l + \sum_j |v_j|_i^{simple} \le |w|_i^{cr} + |uv|_i^{cr} + 1.$$

The case when the split in ω occurs in, immediately before, or immediately after u_p^{-1} is considered similarly.

This proves the upper bound, and finishes the proof.

□

4.4. **Properties of Full i-Length.**

We now consider properties of full i-length – that is, how i-length behaves for a basis.

Lemma 4.18. *For any basis $\mathbf{x}$ of F_n, any $x \in \mathbf{x}$, and any subword w of an i-chunk of $\alpha_{\mathbf{x}}x$, we have $|w|_i^{cr} = 0$.*

It follows that this result holds for any subset of any basis as well.

Proof. Throughout this proof, for sake of simplicity of notation, we write α for $\alpha_{\mathbf{x}}$. As $\mathbf{x}$ is a basis, so is $\alpha\mathbf{x}$. By definition, the i-length of an element or of a set of elements is invariant under conjugation, where we may even conjugate different elements in the set by different conjugators. Therefore cyclic reduction of all elements of $\alpha\mathbf{x}$ does not change any i-length involved. Let $\mathbf{y}$ be the set $\widetilde{\alpha\mathbf{x}}$ obtained from $\alpha\mathbf{x}$ by cyclically reducing every element. Since $\alpha\mathbf{x}$ is a basis, $\mathbf{y}$ is a separable set. Therefore by Theorem 4.3 the augmented Whitehead graph $\hat{\Gamma}_{\mathbf{a}}(\mathbf{y})$ of $\mathbf{y}$ has a cut vertex. Note that this graph does not have vertex loops since each word in $\mathbf{y}$ is cyclically reduced.

Proof by contradiction: assume that there exists some subword w of an i-chunk of αx with $|w|_i^{cr} > 0$. As $|w|_i^{cr} > 0$, by Lemma 4.15, the subgraph Γ' of $\hat{\Gamma}_{\mathbf{a}}(\mathbf{y})$ on the vertex set corresponding to $\mathbf{a} - \{a_i\}$ has no cut vertex (since there are no vertex loops in the graph). It remains to consider the vertices corresponding to $a_i^{\pm 1}$ in $\hat{\Gamma}_{\mathbf{a}}(\mathbf{y})$. Since a_i must appear as a letter in $\mathbf{x}$ and, hence, in $\mathbf{y}$, by the definition of Whitehead graph each of a_i, a_i^{-1} has at least one neighbor in $\hat{\Gamma}_{\mathbf{a}}(\mathbf{y})$.

Consider the case when either a_i or a_i^{-1} has exactly one neighbor in $\hat{\Gamma}_{\mathbf{a}}(\mathbf{y})$. Without loss of generality, assume a_i has exactly one neighbor. If the neighbor b of a_i were in Γ', we would contradict the definition of $w_R(\mathbf{x})$: b^{-1} should have been appended to $w_R(\mathbf{x})$.

Thus, the only neighbor of a_i must be a_i^{-1}. In this case, each occurrence of a (resp. a^{-1}) in $\alpha\mathbf{x}$ must be cyclically followed (resp. preceded) by a (resp. a^{-1}). The only way for this to occur is if every element of $\mathbf{y}$ involving a_i is some power of a_i. But elements of $\mathbf{y}$ are primitives in F_n as they are conjugates of basis elements of $\alpha\mathbf{x}$. Therefore this power can only be $a_i^{\pm 1}$. Moreover, if there are two elements in $\mathbf{y}$ of the form $a_i^{\pm 1}$, then there should be two conjugates of a or a^{-1} in $\alpha\mathbf{x}$, which is impossible because in this case we can obtain a commutator as a primitive element of F_n. Thus, we may assume without loss of generality that a_i is an element of $\mathbf{y}$ and no other element of $\mathbf{y}$ contains an occurrence of a_i.

Since $\mathbf{y}$ was obtained from $\alpha\mathbf{x}$ by conjugating its elements, the structure of $\alpha\mathbf{x}$ is as follows. There is one element of the form a_i^w for some $w \in F_n$, whose conjugate in $\mathbf{y}$ is a_i. All other

elements in $\alpha\mathbf{x}$ are conjugates of words in $\mathbf{y}$ not involving a_i by conjugators that may generally contain a_i. Then $\mathbf{z} = (\alpha\mathbf{x})^{w^{-1}}$ is a basis for F_n one of whose elements is a_i and the others are conjugates of words in $\mathbf{y}$ where the words in $\mathbf{y}$ do not involve a_i (but the conjugators could).

By Proposition 2.3 there is a sequence $(\delta_j), 1 \leq j \leq t$ of elementary Nielsen transformations taking $\mathbf{z}$ to the standard basis $\mathbf{a}$ obtained from the Nielsen reduction process. In other words,

$$(\mathbf{z})\Big(\prod_{j=1}^{t}\delta_j\Big) = \mathbf{a}.$$

Since the Nielsen reduction process does not increase the length of basis elements, the element a_i in $\mathbf{z}$ will be invariant under each transvection δ_j. Let $S = \{j : \ \delta_j \text{ does not involve } a_i\}$ and consider the basis

$$\mathbf{u} = (\mathbf{a})\Big(\prod_{j\in S}\delta_j\Big)^{-1}$$

for F_n. By construction this basis is obtained from $\mathbf{z}$ by removing all occurrences of a_i from $\mathbf{z}$ except a single occurrence of a_i as an element of $\mathbf{z}$. This implies that all other elements of $\mathbf{u}$ form a basis for $\langle \mathbf{a} - \{a_i\}\rangle$. On the other hand, elements of the basis $\mathbf{u}$ are conjugates of elements of $\mathbf{z}$. Therefore cyclic reduction of elements in $\mathbf{u}$ gives the set $\mathbf{y}$ up to cyclic conjugation. But then $\mathbf{y} - \{a_i\}$ is a separable set in $\langle \mathbf{a} - \{a_i\}\rangle$, and is such that $\hat{\Gamma}_{\mathbf{a}-\{a_i\}}(\mathbf{y} - \{a_i\})$ has no cut vertex. This contradicts Theorem 4.3, and shows that neither a_i nor a_i^{-1} may have exactly one neighbor in $\hat{\Gamma}_{\mathbf{a}}(\mathbf{y})$.

We are left to consider the remaining case, when both a_i and a_i^{-1} have at least two neighbors in $\hat{\Gamma}_{\mathbf{a}}(\mathbf{y})$. As Γ' contains no cut vertex, the only way for $\hat{\Gamma}_{\mathbf{a}}(\mathbf{y})$ to still have a cut vertex in this situation is if a_i and a_i^{-1} both have exactly two neighbors in $\hat{\Gamma}_{\mathbf{a}}(\mathbf{y})$, both are neighbors of each other, and both share a common third neighbor, say b. This means that every occurrence of a_i^k, $k \neq 0$, in $\alpha\mathbf{x}$ appears by itself in $\alpha\mathbf{x}$ or appears conjugated by b^{-1}. But this contradicts the definition of $w_C(\mathbf{x})$: the letter b^{-1} should have been appended to w_C. □

As corollaries of the above lemma and definition of the i-length of a set we get the following statements.

Corollary 4.19. *For any basis $\mathbf{x}$ of F_n and any $x \in \mathbf{x}$, $k(\alpha_x x) \leq |\alpha_{\mathbf{x}} x|_i = 0$, and so*

$$|\mathbf{x}|_i = |w_R(\mathbf{x})w_L(\mathbf{x})|_i^{cr}.$$

Lemma 4.20. *For any basis $\mathbf{x}$ and any $x \in \mathbf{x}$ containing a_i,*

$$|\mathbf{x}|_i - 2 \leq |x|_i \leq |\mathbf{x}|_i + 2.$$

Proof. By Lemma 4.18, for any $x \in \mathbf{x}$, $|\alpha_{\mathbf{x}} x|_i^{cr} = 0$. Without loss of generality, as i-length is unaffected by conjugation assume that x is such that all of x, $\alpha'_{\mathbf{x}}x$, and $\alpha_{\mathbf{x}}x$ are cyclically reduced. For simplicity of notation, let $\alpha' := \alpha'_{\mathbf{x}}$. Note that every occurrence of a_i in $\alpha' x$ occurs in a subword of $\alpha' x$ in at least one of the following four forms: a_i, $w_C(\mathbf{x})^{-1}a_i$, $a_i w_C(\mathbf{x})$, $w_C(\mathbf{x})^{-1}a_i w_C(\mathbf{x})$. Similarly, every occurrence of a_i^{-1} in $\alpha' x$ occurs in a subword of $\alpha' x$ in at least one of the forms: a_i^{-1}, $w_C(\mathbf{x})^{-1}a_i^{-1}$, $a_i^{-1}w_C(\mathbf{x})$, $w_C(\mathbf{x})^{-1}a_i^{-1}w_C(\mathbf{x})$.

Consider the following possible cases.

Case 1. The word $\alpha' x$ is a power of a_i, so that no occurrence of a_i (or its inverse) appears in x multiplied by $w_C(\mathbf{x})$ (or its inverse). In this case $w_L(x) = w_R(\mathbf{x})w_L(\mathbf{x})$ and $w_R(x)$ is trivial, therefore $w_R(x)w_L(x) = w_R(\mathbf{x})w_L(\mathbf{x})$ and $|x|_i = |\mathbf{x}|_i$ by Corollary 4.19.

Case 2. Some occurrences of a_i (or its inverse) in $\alpha' x$ occur in subwords of $\alpha' x$ of the form $w_C(\mathbf{x})^{-1}a_i w_C(\mathbf{x})$ (resp. $w_C(\mathbf{x})^{-1}a_i^{-1}w_C(\mathbf{x})$), while some do not. Note that the last letter in $w_C(\mathbf{x})$ must differ from the last letter in $w_R(\mathbf{x})$ since these letters do not cancel in $w_R(\mathbf{x})w_C(\mathbf{x})^{-1}$. But this implies that $w_L(x) = w_L(\mathbf{x})$ and $w_R(x) = w_R(\mathbf{x})$, so $k(x) = k(\mathbf{x})$, again yielding $|x|_i = |\mathbf{x}|_i$.

Case 3. Every a_i (resp. a_i^{-1}) in x occurs in $\alpha' x$ in a subword of $\alpha' x$ of the form $w_C(\mathbf{x})^{-1}a_i w_C(\mathbf{x})$ (resp. $w_C(\mathbf{x})^{-1}a_i^{-1}w_C(\mathbf{x})$). Then $w_L(x)$ contains $w_C(\mathbf{x})^{-1}w_L(\mathbf{x})$ as a terminal segment. It may also contain some portion w_2 of an i-chunk of $\alpha_{\mathbf{x}}(x)$ of zero i-length by Lemma 4.18, and finally it may contain some portion of $w_R(\mathbf{x})w_C(\mathbf{x})$. In any case $w_R(x)$ will contain the rest of $w_R(\mathbf{x})w_C(\mathbf{x})$ and possibly some portion w_1 of an i-chunk of $\alpha_{\mathbf{x}}(x)$ also of zero i-length. Therefore

$$w_R(x)w_L(x) = w_R(\mathbf{x})w_C(\mathbf{x})w_1w_2w_C(\mathbf{x})^{-1}w_L(\mathbf{x}),$$

where w_1 and w_2 may be trivial. It follows that $k(x) \le k(\mathbf{x}) = 0$, so $|x|_i = |w_R(x)w_L(x)|_i^{cr}$. Therefore applying Lemmas 4.17 and 4.16, and taking into account that conjugation does not change the conjugate reduced i-length of a subword, we obtain

$$\begin{aligned}
|x|_i = |w_R(x)w_L(x)|_i^{cr} &= |w_R(\mathbf{x})w_C(\mathbf{x})^{-1}w_1w_2w_C(\mathbf{x})w_L(\mathbf{x})|_i^{cr}\\
&\le |w_R(\mathbf{x})w_L(\mathbf{x})|_i^{cr} + |w_C(\mathbf{x})^{-1}w_1w_2w_C(\mathbf{x})|_i^{cr} + 1\\
&= |w_R(\mathbf{x})w_L(\mathbf{x})|_i^{cr} + |w_1w_2|_i^{cr} + 1\\
&\le |w_R(\mathbf{x})w_L(\mathbf{x})|_i^{cr} + |w_1|_i^{cr} + |w_2|_i^{cr} + 2\\
&= |w_R(\mathbf{x})w_L(\mathbf{x})|_i^{cr} + 2 = |\mathbf{x}|_i + 2.
\end{aligned}$$

Similarly we derive the first inequality

$$\begin{aligned}
|\mathbf{x}|_i = |w_R(\mathbf{x})w_L(\mathbf{x})|_i^{cr} &= |w_R(\mathbf{x})w_C(\mathbf{x})^{-1}w_1w_2(w_1w_2)^{-1}w_C(\mathbf{x})w_L(\mathbf{x})|_i^{cr}\\
&\le |w_R(\mathbf{x})w_C(\mathbf{x})^{-1}w_1w_2w_C(\mathbf{x})w_L(\mathbf{x})|_i^{cr} + |w_1w_2|_i^{cr} + 1\\
&\le |w_R(x)w_L(x)|_i^{cr} + |w_1|_i^{cr} + |w_2|_i^{cr} + 2\\
&= |w_R(x)w_L(x)|_i^{cr} + 2 = |x|_i + 2.
\end{aligned}$$

Note that the last case covers all situations when $w_C(\mathbf{x})$ is trivial.

□

As an immediate corollary we obtain the following statement.

Corollary 4.21. *For any bases* $\mathbf{x}$ *and* $\mathbf{y}$ *sharing a common element containing* a_i*,*

$$\big|\, |\mathbf{x}|_i - |\mathbf{y}|_i \big| \le 4.$$

Proof. Let x be a common element of $\mathbf{x}$ and $\mathbf{y}$ containing a_i. Then by Lemma 4.20 we get

$$\big|\, |x|_i - |\mathbf{x}|_i \big| \le 2,$$

$$\big|\, |x|_i - |\mathbf{y}|_i \big| \le 2.$$

Combining the above inequalities proves the lemma. □

The following corollary is not used in the rest of the paper, but is an interesting observation on its own.

Corollary 4.22. *For any basis* $\mathbf{x}$ *of* F_n *there is at most one* $i \in \{1, 2, \dots, n\}$ *such that* $|\mathbf{x}|_i > 0$*.*

Proof. Suppose that $|\mathbf{x}|_i > 0$ for some i. Let $\tilde{\mathbf{x}}$ be the separable set obtained from $\mathbf{x}$ by cyclically reducing all of its elements. Consider the augmented Whitehead graph $\hat{\Gamma}$ of $\tilde{\mathbf{x}}$. By construction, $\hat{\Gamma}$ includes the Whitehead graphs of all i-chunks of all elements of $\tilde{\mathbf{x}}$ as subgraphs. Since by definition of full i-length, $|\tilde{\mathbf{x}}|_i = |\mathbf{x}|_i > 0$ we must have at least one i-chunk w of $\tilde{\mathbf{x}}$ with $|w|_i^{cr} > 0$, which implies by Lemma 4.15 that corresponding Whitehead graph $\Gamma_i = \Gamma_{\mathbf{a}-a_i}(w)$ does not have a cut vertex. But according to Theorem 4.3 graph $\hat{\Gamma}$ has a cut vertex.

As Γ_i is a subgraph of $\hat{\Gamma}$, it must be that removing any cut vertex v of $\hat{\Gamma}$ creates at least two connected components, with $\Gamma_i - \{v\}$ in one component and at least one of a_i or a_i^{-1} in a different component. Without loss of generality, say a_i appears in a different component. Then a_i can have at most one neighbor in $\Gamma_i \subset \hat{\Gamma}$. If a_i has no such neighbor, then either $\{a_i\}$ or $\{a_i, a_i^{-1}\}$ will be a connected component in $\hat{\Gamma}$. Therefore either $\{a_i\}$ or $\{a_i, a_i^{-1}\}$ will be a nontrivial proper connected component in Γ_j for any $j \ne i$. But this means that any vertex in Γ_j, except possibly a_i or a_i^{-1}, will be a cut vertex. By Lemma 4.15 we have $|\mathbf{x}|_j = 0$.

If a_i has a neighbor in $\Gamma_i \subset \hat{\Gamma}$ – without loss of generality, say a_k for some $k \neq i$ – and a_i is not adjacent to a_i^{-1}, then a_i is isolated in Γ_k, and a_k is a cut vertex in Γ_j for $j \neq i, k$. In this case, $|\mathbf{x}|_j = 0$ for $j \neq i$.

If a_i has a neighbor in $\Gamma_i \subset \hat{\Gamma}$ – again, say a_k for $k \neq i$ – and a_i is adjacent to a_i^{-1}, then a_i^{-1} may have no other neighbors in $\Gamma_i \subset \hat{\Gamma}$ except a_k because otherwise $\hat{\Gamma}$ would not have a cut vertex. Here again, either $\{a_i\}$ or $\{a_i, a_i^{-1}\}$ will be a nontrivial proper connected component in Γ_k, so $|\mathbf{x}|_k = 0$. Finally, for every $j \neq i, k$ the vertex a_k will be a cut vertex in Γ_j yielding $|\mathbf{x}|_j = 0$. □

5. The Geometry of $\mathcal{ES}_n^1$

5.1. Distance and i-Length.

We are now ready to estimate distances in $\mathcal{ES}_n^1$, based on how much i-length can change in a single proper nonempty index set.

Lemma 5.1. *For any proper nonempty subset S of the index set $\{1, \dots, n\}$, any basis $\mathbf{x} = \{x_1, \dots, x_n\}$ of F_n, and any S-transformation $\phi \in \mathrm{Aut}(F_n)$ which is the identity on $\mathbf{x}_{\overline{S}}$, we have that*

$$|\mathbf{x}|_i - 12 \leq |\mathbf{x}\phi|_i \leq |\mathbf{x}|_i + 12.$$

Proof. If there exists some $x \in \mathbf{x}_{\overline{S}}$ such that x contains an occurrence of $a_i^{\pm 1}$, then since $x \in \mathbf{x} \cap \mathbf{x}\phi$, by Corollary 4.21

$$\big|\, |\mathbf{x}|_i - |\mathbf{x}\phi|_i \big| \leq 4,$$

and the lemma follows.

If no such x exists, choose any $x \in \mathbf{x}_{\overline{S}}$ and let $y \in \mathbf{x}_S$ be an element of $\mathbf{x}$ which contains an occurrence of $a_i^{\pm 1}$. Let $\mathbf{x}' := (\mathbf{x} - \{x\}) \cup \{xy\}$.

The bases $\mathbf{x}$ and $\mathbf{x}'$ share y in common, so by Corollary 4.21

$$\big|\, |\mathbf{x}|_i - |\mathbf{x}'|_i \big| \leq 4, \tag{6}$$

Since $\mathbf{x}'$ and $\mathbf{x}'\phi$ share xy in common, we get

$$\big|\, |\mathbf{x}'|_i - |\mathbf{x}'\phi|_i \big| \leq 4, \tag{7}$$

Also there must be an element among $z \in (\mathbf{x}\phi)_S$ containing an occurrence of a_i (otherwise $\mathbf{x}\phi$ would not contain an occurrence of a_i). This element z is common for bases $\mathbf{x}\phi$ and $\mathbf{x}'\phi$ yielding

$$\big|\, |\mathbf{x}'\phi|_i - |\mathbf{x}\phi|_i \big| \leq 4, \tag{8}$$

Finally, combining the inequalities (6), (7) and (8), we obtain the statement of the lemma.

□

It seems that, with more careful bookkeeping, the constant 12 might be able to be improved.

Corollary 5.2. *Let $\mathbf{x}$ be a basis of F_n. Then the number of index changes required in a transformation from $\mathrm{Aut}(F_n)$ taking $\mathbf{a}$ to $\mathbf{x}$ is bounded below by $\frac{1}{24}|\mathbf{x}|_i - 1$.*

Proof. For any index set S, an S-transformation can be written as a product of an S-transformation which is identity on $\mathbf{x}_S$ and an S-transformation which is identity on $\mathbf{x}_{\overline{S}}$. Applying Lemma 5.1 twice, we see an S-transformation can change i-length by at most 24. The corollary then follows from Theorem 3.2. Note that the requirement about the compatibility of the neighboring index sets cannot decrease the number of subwords in the optimal decomposition realizing the minimal number of index changes. □

This corollary, combined with Theorem 3.2, shows our main computational theorem:

Theorem 5.3. *Let $\mathbf{x}$ be a basis of F_n, expressed in terms of a fixed standard basis $\mathbf{a}$. For any index i and any index sets S_a and S_x,*

$$d_{\mathcal{ES}_n^1}((\mathbf{a}, S_a), (\mathbf{x}, S_x)) \geq \frac{|\mathbf{x}|_i}{24} - 1.$$

5.2. $\mathcal{ES}^1_n$ **is Not Hyperbolic.**

Corollary 5.2 is useful for estimating distances in $\mathcal{ES}^1_n$. For instance, we may now apply this corollary to show that $\mathcal{ES}^1_n$ is not hyperbolic in the sense of Gromov, by identifying quasiflats – that is, a quasiisometric embedding $\mathbb{R}^k \to \mathcal{ES}^1_n$ for $k > 1$.

Let $p_t := a_1^{t+1}a_2^{t+1}\cdots a_{n-1}^{t+1}a_1^{t+1}a_2^{t+1}a_1^{t+1}$. Note that for $t \geq 1$ the augmented Whitehead graph of p_t looks similar to the graph shown in Figure 3, and removing vertices corresponding to a_n and a_n^{-1} will produce graphs without cut vertices. We propose to map the integer lattice $\mathbb{Z}^m$ quasiisometrically into $\mathcal{ES}^1_n$ by the map ψ which takes $(k_1, k_2, \ldots, k_m) \in \mathbb{Z}^m$ to the vertex $\psi(k_1, k_2, \ldots, k_m) = (\mathbf{x}, S)$ of $\mathcal{ES}^1_n$, where S is an arbitrary proper nontrivial subset of $\{1, 2, \ldots, n\}$ and $\mathbf{x}$ is obtained from the standard basis $\mathbf{a}$ by replacing a_n by $a_n p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$.

Theorem 5.4. *The map ψ yields an m-dimensional quasiflat in $\mathcal{ES}^1_n$.*

Proof. To see that ψ is indeed a quasiisometry, consider the images of two points, $(k_1, k_2, \ldots, k_m)$ and $(l_1, l_2, \ldots, l_m)$ under ψ. In the domain, these points are of distance

$$d = \sum_{t=1}^{m} |k_t - l_t|$$

apart. In the codomain, the distance between $\psi(k_1, k_2, \ldots, k_m)$ and $\psi(l_1, l_2, \ldots, l_m)$ is the same as the distance between the basepoint $\mathbf{a}$ and the point represented by the standard basis with a_n replaced by $a_n\omega$, where

$$\omega = p_m^{-k_m} p_{m-1}^{-k_{m-1}} \cdots p_1^{-k_1} p_1^{l_1} p_2^{l_2} \cdots p_m^{l_m}$$

after free reduction.

By Theorem 5.3 and the definition of full i-length, the latter distance is bounded below by

$$\frac{1}{24}|\omega|_n^{cr} - 1. \tag{9}$$

We claim $|\omega|_n^{cr} \geq \frac{d}{11} - \frac{21}{11}$, as follows. By Lemma 4.12 $|\omega|_n^{cr}$ can be estimated from below by

$$|\omega|_n^{cr} \geq \min_{\mathcal{F}\in S}\left(\max\left\{\frac{|\mathcal{F}|}{2} - 1,\ \frac{1}{5}|\omega - \mathcal{F}|_n^{simple} - 3\right\}\right), \tag{10}$$

where S is the set consisting of all nested families of canceling pairs in ω. Let $\mathcal{F}$ denote the family of canceling pairs in ω that minimizes the bound in (10).

There may be free cancellations of two types in ω. First, several full occurrences of p_t may cancel with full occurrences of p_t^{-1} in the middle where $p_1^{-k_1}$ and $p_1^{l_1}$ meet, and second, there may be cancellation of two occurrences of p_t for different t. In the second case, by the definition of p_t, the only part that may cancel is a subword of either the last and/or the first syllable of the form a_1^{t+1}. However, reductions of the second type preserve Whitehead graphs in the following sense: the Whitehead graph of the uncanceled subword q of every copy of $p_t^{\pm 1}$ (with vertices $a_n^{\pm 1}$ removed) will still have a cut vertex. We will call such a subword q a *leftover of type* t and denote by $q^{(t)}$. Each $q^{(t)}$ contains $a_2^{t+1}\ldots a_{n-1}^{t+1}a_1^{t+1}a_2^{t+1}$ as a subword.

Consider canceling pairs in $\mathcal{F}$. Every occurrence of $q^{(t)}$ disjoint from pairs in $\mathcal{F}$ will introduce 1 to the sum in the definition of $|\omega - \mathcal{F}|_n^{simple}$. Therefore we just need to count the number of such $q^{(t)}$'s to get a lower bound for $|\omega - \mathcal{F}|_n^{simple}$. A canceling pair in $\mathcal{F}$ may contain some $q^{(t)}$ in one factor and $(q^{(t)})^{-1}$ in the other factor. We remove any such $q^{(t)}$ from our count by throwing out all but $|k_t - l_t|$ occurrences of any $q^{(t)}$ for each t. This leaves exactly d possible $q^{(t)}$'s to count. For the remaining $q^{(t)}$'s, by the definition of p_t, a given canceling pair may involve at most 4 different occurrences of a $q^{(t)}$ (for possibly different values of t). Thus, removing all $p_t^{\pm 1}$ that cancel in the free reduction of the first type in the previous paragraph, then all leftovers that are contained in a canceling pair which also contains the inverse of the leftover, and finally removing all leftovers that intersect a canceling pair at all will leave at least $d - 4|\mathcal{F}|$ occurrences of a $q^{(t)}$.

Hence,

$$\frac{1}{5}|\omega - \mathcal{F}|_n^{simple} - 3 \geq \frac{1}{5}\big((d - 4|\mathcal{F}|) + |\mathcal{F}|\big) - 3 = \frac{d}{5} - \frac{3}{5}|\mathcal{F}| - 3.$$

From (10) we obtain

$$|\omega|_n^{cr} \geq \max\left\{\frac{|\mathcal{F}|}{2} - 1,\ \frac{d}{5} - \frac{3}{5}|\mathcal{F}| - 3\right\},$$

If $|\mathcal{F}| \geq \frac{2}{11}d - \frac{20}{11}$ then

$$|\omega|_n^{cr} \geq \frac{|\mathcal{F}|}{2} - 1 \geq \frac{d}{11} - \frac{21}{11}.$$

But if $|\mathcal{F}| < \frac{2}{11}d - \frac{20}{11}$ then

$$|\omega|_n^{cr} \geq \frac{d}{5} - \frac{3}{5}|\mathcal{F}| - 3 > \frac{d}{11} - \frac{21}{11}.$$

In either case, as claimed, $|\omega|_n^{cr} \geq \frac{d}{11} - \frac{21}{11}$.

Combining this claim with the lower bound in (9), we have that the distance between the vertices $\psi(k_1, k_2, \ldots, k_m)$ and $\psi(l_1, l_2, \ldots, l_m)$ is bounded below by $\frac{d}{24\cdot 11} - \frac{21}{24\cdot 11} - 1 = \frac{1}{264}d - \frac{95}{88}$.

We claim the distance between $\psi(k_1, k_2, \ldots, k_m)$ and $\psi(l_1, l_2, \ldots, l_m)$ is also bounded above by $d+m$, as follows. Without loss of generality, fix $S = \{n\}$. Recall a vertex in $\mathcal{ES}_n^1$ is defined up to conjugation. Thus, for any word w,

$$[\langle a_1, \ldots, a_{n-1}\rangle * \langle a_n\rangle] = [\langle a_1^w, \ldots, a_{n-1}^w\rangle * \langle a_n^w\rangle].$$

Thus, the following describes a path from $\psi(k_1, k_2, \ldots, k_m)$ to $\psi(l_1, l_2, \ldots, l_m)$:

$$\begin{array}{rcl}
\psi(k_1, k_2, \ldots, k_m) & = & [\langle a_1, \ldots, a_{n-1}\rangle * \langle a_n p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}\rangle] \\
& = & [\langle a_1, \ldots, a_{n-1}\rangle * \langle p_1^{-k_1} \ldots p_m^{-k_m} a_n\rangle] \\
& \to & [\langle a_1, \ldots, a_{n-1}\rangle * \langle p_1^{-k_1} \ldots p_m^{-k_m} a_n p_1^{l_1-k_1}\rangle] \\
& = & [\langle a_1, \ldots, a_{n-1}\rangle * \langle p_2^{-k_2} \ldots p_m^{-k_m} a_n p_1^{l_1}\rangle] \\
& \to & [\langle a_1, \ldots, a_{n-1}\rangle * \langle p_2^{-k_2} \ldots p_m^{-k_m} a_n p_1^{l_1} p_2^{l_2-k_2}\rangle] \\
& = & [\langle a_1, \ldots, a_{n-1}\rangle * \langle p_3^{-k_3} \ldots p_m^{-k_m} a_n p_1^{l_1} p_2^{l_2}\rangle] \\
& \to & \ldots \\
& \to & [\langle a_1, \ldots, a_{n-1}\rangle * \langle p_m^{-k_m} a_n p_1^{l_1} p_2^{l_2} \ldots p_{m-1}^{l_{m-1}} p_m^{l_m-k_m}\rangle] \\
& = & [\langle a_1, \ldots, a_{n-1}\rangle * \langle a_n p_1^{l_1} p_2^{l_2} \ldots p_{m-1}^{l_{m-1}} p_m^{l_m}\rangle] \\
& = & \psi(l_1, l_2, \ldots, l_m).
\end{array}$$

At each arrow, the above sequence is the same: for some integers i and j, we append p_i^j to the last factor in a vertex of the form $[\langle a_1, \ldots, a_{n-1}\rangle * \langle uav\rangle]$. We claim this may be done in via at most $2j+1$ steps in $\mathcal{ES}_n^1$, by the following edge path, where in this sequence, arrows each represent crossing exactly 1 edge of $\mathcal{ES}_n^1$:

$$\begin{array}{rl}
& [\langle a_1, \ldots, a_{n-1}\rangle * \langle uav\rangle] \\
\to & [\langle a_{n-1}\rangle * \langle a_1, \ldots, a_{n-2}, ua_nv\rangle] \\
= & [\langle a_{n-1}\rangle * \langle a_1, \ldots, a_{n-2}, ua_nva_1^{i+1}a_2^{i+1} \cdots a_{n-2}^{i+1}\rangle] \\
\to & [\langle a_3, \ldots, a_{n-2}\rangle * \langle a_1, a_2, a_{n-1}, ua_nva_1^{i+1}a_2^{i+1} \cdots a_{n-2}^{i+1}\rangle] \\
= & [\langle a_3, \ldots, a_{n-2}\rangle * \langle a_1, a_2, a_{n-1}, ua_nvp_i\rangle] \\
\to & [\langle a_{n-1}\rangle * \langle a_1, \ldots, a_{n-2}, ua_nvp_i\rangle] \\
= & [\langle a_{n-1}\rangle * \langle a_1, \ldots, a_{n-2}, ua_nvp_ia_1^{i+1}a_2^{i+1} \cdots a_{n-2}^{i+1}\rangle] \\
\to & [\langle a_3, \ldots, a_{n-2}\rangle * \langle a_1, a_2, a_{n-1}, ua_nvp_ia_1^{i+1}a_2^{i+1} \cdots a_{n-2}^{i+1}\rangle] \\
= & [\langle a_3, \ldots, a_{n-2}\rangle * \langle a_1, a_2, a_{n-1}, ua_nvp_i^2\rangle] \\
\to & \ldots \\
= & [\langle a_3, \ldots, a_{n-2}\rangle * \langle a_1, a_2, a_{n-1}, ua_nvp_i^j\rangle] \\
\to & [\langle a_1, \ldots, a_{n-1}\rangle * \langle uavp_i^j\rangle]
\end{array}$$

Note here we shown the path when $j > 0$; the path when $j < 0$ is similar. Combining these two descriptions, we have that:

$$d_{\mathcal{ES}_n^1}(\psi(k_1, k_2, \ldots, k_m), \psi(l_1, l_2, \ldots, l_m)) \leq \sum_{i=1}^{m} 2|l_i - k_i| + 1 = 2d + m.$$

As distances are bounded both above and below, we have a quasiisometry. □

As immediate corollaries, we obtain:

Corollary 5.5. *The graph $\mathcal{ES}_n^1$ is not hyperbolic in the sense of Gromov.*

This shows that $\mathcal{ES}_n^1$ does not have the hyperbolicity desired for an analogue for $\mathrm{Out}(F_n)$ of the curve complex for the mapping class group. The hyperbolicity of the curve complex was shown by Masur and Minsky [MM99], and has proven to be useful in numerous situations.

Corollary 5.6. *The space $\mathcal{ES}_n^1$ has infinite asymptotic dimension. The dimension of every asymptotic cone of $\mathcal{ES}_n^1$ is infinite.*

Corollary 5.7. *The identity map on vertices between $\mathcal{ES}_n^1$ and $\mathcal{FF}_n^1$ is not a quasiisometry. Moreover, there is no coarsely* $\mathrm{Out}(F_n)$*-equivariant quasiisometry between $\mathcal{ES}_n^1$ and $\mathcal{FF}_n^1$.*

Proof. The first half follows immediately from Theorem 5.4, as the set $\psi\mathbb{Z}^m \subset \mathcal{ES}_n^1$ has diameter 1 in $\mathcal{FF}_n^1$: for $k \neq n$, the element a_k has translation length 0 on the Bass-Serre tree of every element in $\psi\mathbb{Z}^m$.

For the second half we note that since $\mathrm{Out}(F_n)$ acts on both $\mathcal{ES}_n^1$ (and $\mathcal{FF}_n^1$) by isometries, for each $\phi \in \mathrm{Out}(F_n)$ the orbits under powers of ϕ of vertices of $\mathcal{ES}_n^1$ (and $\mathcal{FF}_n^1$) are either all bounded or all unbounded. Now consider $\phi \in \mathrm{Out}(F_n)$ taking the standard basis $\mathbf{a}$ of F_n to the basis obtained from $\mathbf{a}$ by replacing a_n with $a_n p_1$. By construction, for each index set S the orbit of $(\mathbf{a}, S)$ in $\mathcal{ES}_n^1$ under iterations of ϕ is unbounded as the i-length of $\phi^n(\mathbf{a})$ grows. But the same orbit is bounded in $\mathcal{FF}_n^1$. Thus, there is no coarsely $\mathrm{Out}(F_n)$-equivariant quasiisometry between $\mathcal{ES}_n^1$ and $\mathcal{FF}_n^1$ because every orbit in $\mathcal{FF}_n^1$ under iterations of ϕ is bounded and cannot be an image of an unbounded orbit in $\mathcal{ES}_n^1$. □

An analogous results with identical proofs hold true for the relationships between the free factorization graph $\mathcal{ES}_n^1$ and the free factor graph $\mathcal{F}_n^1$ and between $\mathcal{ES}_n^1$ and the free splitting graph $\mathcal{FS}_n^1$. There is a natural (coarsely well-defined for $n > 2$) map $\Sigma\colon \mathcal{ES}_n^1 \to \mathcal{F}_n^1$ defined by sending a vertex $[A * B]$ in $\mathcal{ES}_n^1$ to the vertex $[A]$ in $\mathcal{F}_n^1$. This map is induced by the same map on vertices from $\mathcal{FF}_n^1$ to $\mathcal{F}_n^1$, which is a coarsely $\mathrm{Out}(F_n)$-equivariant quasiisometry. Also there is a natural embedding $\imath\colon \mathcal{ES}_n^1 \to \mathcal{FS}_n^1$ defined by sending a vertex $[A * B]$ in $\mathcal{ES}_n^1$ to the vertex $[A * B]$ in $\mathcal{FS}_n^1$, which is quasisurjection. However, neither of the above maps is a quasiisometry.

Corollary 5.8. *The maps $\Sigma\colon \mathcal{ES}_n^1 \to \mathcal{F}_n^1$ and $\imath\colon \mathcal{ES}_n^1 \to \mathcal{FS}_n^1$ are not quasiisometries. Moreover, there is no coarsely* $\mathrm{Out}(F_n)$*-equivariant quasiisometry between $\mathcal{ES}_n^1$ and $\mathcal{F}_n^1$, and between $\mathcal{ES}_n^1$ and $\mathcal{FS}_n^1$.*

The last corollary provides a negative answer to a question of Bestvina and Feighn (the first half of Question 4.4 in [BF10]).

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